An ℓ_2-Consistent Event-Triggered Control Policy for Linear Systems with Output Feedback

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***Abstract*— In this article, we consider a control loop where a scheduler collocated with the sensors decides when to transmit the measured output to the controller with the objective of saving valuable communication resources, while still meeting desirable closed-loop stability and performance requirements. The sensors provide, in general, only measured outputs of the system. The performance of the system is defined as the ℓ_2-gain between an exogenous disturbance input and a performance output of interest. When the scheduler triggers transmissions periodically, one can determine the controller that results in a given ℓ_2-gain bound along with conditions for its existence by following a game-theoretical approach. Based on these results, we tackle the problem of designing a scheduler-controller pair that guarantees the same ℓ_2-gain bound as the designed periodic time-triggered ℓ_2-controller, however, with a lower or at most the same average transmission rate from the sensors to the controller. We refer to this property as ℓ_2-consistency and thus we will provide such an ℓ_2-consistent controller-scheduler pair. Numerical examples show significant communication savings for typical disturbance inputs, while guaranteeing the same ℓ_2-gain bound for the control loop as the periodic solution.**

I INTRODUCTION

Ensuring safe and reliable operation of control systems in the presence of unwanted disturbances is crucial in many control applications in areas such as automotive, aerospace, healthcare, power generation and many others. In many of these applications no probabilistic information about the disturbances is available and the system has to be designed to cope with all possible disturbances, including worst-case scenarios [1]. This usually leads to conservative designs and often a particularly large waste of computation and communication resources. A prime example is the choice of conservatively large sampling and transmission rates to ensure that signals flow rapidly among sensors, controllers and actuators so that the performance requirements are met for all possible disturbances; thereby the whole control infrastructure is configured based on worst-case scenarios, while it is intended for typical (common) scenarios. This calls for more resource-aware and more flexible computation and communication strategies that adapt based on the actual disturbance scenarios encountered. In this article we will focus on such designs in the context of ℓ_2-control.

The ℓ_2-control (or H_∞) framework has been widely used since the 90s in order to ensure that performance specifications are met for all possible disturbances within a certain class. In fact, within this framework, considering a discrete-time linear system, with output measurement, additive disturbances and a performance output of interest, one can assert if, and design a controller such that, the ℓ_2-attenuation level from the disturbances to the performance output is bounded by a prescribed value for *all* squared summable disturbances. The infimum value of this attenuation level is called the ℓ_2-gain of the system. The problem that we tackle in this article, is to understand if by using a different communication scheme (other than periodic) one can provide the same guarantees, but with a smaller average transmission rate. In particular, we do this for the setup depicted in Fig. 1, where G, S, K and N represent the discrete-time plant, the transmission scheduler, the controller and the communication network between the scheduler and the controller, respectively. The goal of the joint scheduling and ℓ_2-control policy is to attenuate the effect of the disturbance inputs w on the performance output z of the system. This problem can be positioned in the area of event-triggered control (ETC), see, e.g. the overviews [2], [3], which tackles the problem of designing appropriate (state or output-dependent) schedulers to arbitrate data transmission from the sensors to the controller.

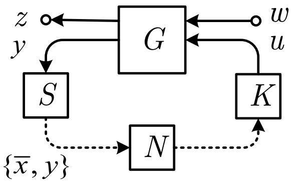

Fig. 1: The output feedback ℓ_2-controller when communication takes place over a resource-constraint communication network. G, K, S and N refer to the plant, the controller, the scheduler and the network, respectively. The state estimation in the scheduler is denoted by $\bar{x}$.

We investigated this problem in the framework of full-state feedback in [4], [5] and here we focus on the more challenging framework of output feedback. We start by noticing that, when the scheduler triggers transmissions periodically, one can determine the controller that results in a given ℓ_2-gain bound along with conditions for its existence by following a game-theoretical approach. Using these results, we explore the design of an ETC, borrowing the notion of "ℓ_2-consistent ETC" from our previous works [4], [5]. This consistency concept refers to any ETC that preserves the same ℓ_2-gain bound as a periodic time-triggered ℓ_2-controller, however, with a lower or at most equal average transmission rate. The proposed event-triggered scheduling law in [5] depends explicitly on the disturbance input, which can only be calculated if the state vector is fully measured at all time steps. However, when the sensors only measure an output, rather than the full-state, then the calculation of the disturbance inputs is, in general, not possible. The main contribution of this work is to propose an ℓ_2-consistent ETC for this challenging output feedback case.

Related work: In the ETC literature one can find a considerable amount of research that considers an average quadratic cost, which can jointly capture the effects of the average tracking errors and the control efforts as a performance index. Finding the optimal joint event-triggered scheduler and controller that minimize such a performance index, related to H_2 control, is one of the problems of interest in the area [6], [7]. However, since this problem is in general hard, the research in this domain is inclined mainly toward the design of ETCs, which outperform periodic time-triggered controls (PTCs) in the sense of an average quadratic cost [8]–[11]. An ETC which can guarantee a better trade-off between the average quadratic performance and the average transmission rate in comparison to the optimal PTC is called consistent [9]. Researchers also investigate the design of ETCs in the context of $\mathcal{L}_\infty$-gain and $\mathcal{L}_p$-gain in recent years, see, e.g., [12]–[14]. As an example, a class of dynamic ETCs is introduced in [13] that provides a trade-off between inter-event times, maximum allowable delays and $\mathcal{L}_p$-gains. Furthermore, the design of ETCs from an ℓ_2 (or $\mathcal{L}_2$)-gain perspective has received a great deal of attention throughout the past decade, see, e.g., [15]–[19]. In most of these studies a Lyapunov-based approach is followed to design ETCs, which can provide a guaranteed bound for the ℓ_2 (or $\mathcal{L}_2$)-gain of the system. However, a clear comparison between the average transmission rate of the proposed ETC and its periodic time-triggered ℓ_2-controller counterpart, which guarantees the same ℓ_2-gain bound for the closed-loop system, is often absent. Therefore, we extend the notion of consistency for the event-triggered ℓ_2-controllers, which simultaneously takes into account an ℓ_2-gain bound of the control loop and the average data transmission rate from the sensors to the controller.

We are just aware of one other research paper [20], where the proposed ETC has the same consistency property as we define in the present work. Both [20] and [11] build upon the results in [21] which proposes a constructive algorithm to redesign optimal H_2 and H_∞ controllers, such that sampled-data solutions are still optimal under any sampling pattern. This approach is radically different in the H_2 setting [11] from the approach in the mentioned works that rely on dynamic programming [6], [9], [10] . In particular it does not take into account the information that is present when a transmission does not occur [6]. Our approach is in the spirit of [22] where the min-max algorithm replaces dynamic programming to account for worst-case disturbances. Since [20] is presented in continuous-time and follows a radically different approach, connections between the results are hard to establish. Both works are therefore of independent interest.

Organization: The remainder of this paper is organized as follows. Firstly, a periodic time-triggered ℓ_2-controller is designed in Section II, where the problem of interest is also introduced. An ℓ_2-consistent ETC is proposed in Section III, which is the main contribution of this work. The effectiveness of the proposed ETC in decreasing the communication rate is demonstrated through a numerical example in Section IV. Section V presents some concluding remarks. The proofs of lemmas and theorems can be found in the appendix.

Notation: For $r, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define $\mathbb{N}_r^s = \{t \in \mathbb{N}_0 | r \leqslant t \leqslant s\}$ and $\ell^d_{2,r,s}$ as the Hilbert space of sequences $w_{r:s} := \{w_t\}_{t \in \mathbb{N}_r^s}$, where $w_t \in \mathbb{R}^d$ for all $t \in \mathbb{N}_0$. The ℓ_2-norm of $w_{r:s} \in \ell^d_{2,r:s}$ is given by $\|w_{r:s}\|_{\ell_2} := \sqrt{\sum_{t=r}^{s} \|w_t\|^2}$, where $\|w_t\|^2 = w_t^\intercal w_t$. We also define $\|w_t\|_Q^2 := w_t^\intercal Q w_t$ for every square matrix Q with suitable dimension. Moreover, ℓ_2^d is the Hilbert space of square summable sequences $w = \{w_t\}_{t \in \mathbb{N}_0}$ with norm $\|w\|_{\ell_2} := \sqrt{\sum_{t \in \mathbb{N}_0} \|w_t\|^2}$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be upper semicontinuous if $f(x_0) \geqslant \limsup_{x \to x_0} f(x)$ at all $x_0 \in \mathbb{R}^n$.

II Problem formulation

Section II-A introduces the problem setting with periodic communication and Sections II-C, II-D provide controllers that guarantee an ℓ_2-gain bound with periodic communication, relying on the notion of information state, introduced in Section II-B. In Section II-E, we introduce the structure of the NCS, when the transmissions follow an event-triggered policy instead of a periodic one. Finally, we present the problem of interest in Section II-F. Throughout the article, the most important equations of each Section, Lemma or Definition are highlighted. This can help readers to quickly grasps the main messages.

II-A Networked control system with periodic communication

Consider the system architecture in Fig. 1 in which the plant G is given by a discrete-time linear time-invariant (LTI) system

$$\begin{aligned} x_{t+1} &= A x_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t, \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^o$, $z_t \in \mathbb{R}^p$, $u_t \in \mathbb{R}^m$ and $w_t \in \mathbb{R}^d$ are the state, the measured output, the performance output, the control input and the disturbance, respectively, at discrete time $t \in \mathbb{N}_0$. A, B_1, B_2, C_1, D_{12}, C_2, and D_{21} are real matrices of appropriate dimensions and $w \in \ell_2^d$. Moreover, we define $\delta_t \in \{0, 1\}$ to indicate whether communication takes place from sensors to controller at time $t \in \mathbb{N}_0$ ($\delta_t = 1$) or not ($\delta_t = 0$). If $\delta_t = 1$, all the measured outputs $(y_{\mathsf{L}_{t-1}+1}, y_{\mathsf{L}_{t-1}+2}, \ldots, y_t)$ after the last transmission time (before t) $\mathsf{L}_{t-1} := \max\{\ell \in \mathbb{N}_0^{t-1} | \delta_\ell = 1\}$ up to (and including) the current time step t are transmitted. When the scheduler follows a periodic transmission policy with a given time period $\tau \in \mathbb{N}$, we set $\delta_t = \pi_t^\tau$ at all $t \in \mathbb{N}_0$, where

$$\pi_t^\tau := \begin{cases} 1, & \text{if } t \text{ is zero or an integer multiple of } \tau \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then, any periodic control policy can be formulated as

$$u_t = \mathcal{R}_t^{\pi_\tau}(\mathcal{F}_t^{\pi_\tau}), \tag{3}$$

for a mapping $\mathcal{R}_t^{\pi_\tau} : \mathcal{F}_t^{\pi_\tau} \to \mathbb{R}^m$, where

$$\mathcal{F}_t^{\pi_\tau} := \{y_i | i \in \mathbb{N}_0^{\mathsf{L}_t}\} \tag{4}$$

is the information set available for the controller at every $t \in \mathbb{N}_0$. We need the definition of global asymptotic stability in the sequel.

Definition 1: (Global asymptotic stability) The system (1) with $w = (0, 0, \dots)$ and a given controller is said to be globally asymptotically stable (at equilibrium point $x_e = 0$), if:

i) it is Lyapunov stable, i.e., for every $\zeta \succ 0$, there exists an $\epsilon \succ 0$ such that for all initial states $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leqslant \epsilon$, it holds for the corresponding state trajectory $\{x_t\}_{t \in \mathbb{N}_0}$ that $\|x_t\| \leqslant \zeta$ for every $t \in \mathbb{N}_0$,

ii) it is globally attractive, i.e., for all initial state $x_0 \in \mathbb{R}^n$, the corresponding state trajectory $\{x_t\}_{t \in \mathbb{N}_0}$ converges to $x_e = 0$ as time goes to infinity, i.e., $\lim_{t \to \infty} x_t = 0$. □

Now we formally define the concept of a τ-periodic ℓ_2-controller for system (1).

Definition 2: (*τ-periodic ℓ_2-controller*) Given $\gamma \in \mathbb{R}_{\succ 0}$ and $\tau \in \mathbb{N}$, a periodic control policy $\mathcal{R}_t^{\pi_\tau} : \mathcal{F}_t^{\pi_\tau} \to \mathbb{R}^m$, $t \in \mathbb{N}_0$, for the system (1), where $\mathcal{F}_t^{\pi_\tau}$ is given in (4), is referred to as a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ [23], if:

i) the control loop given by (1) and (3) is globally asymptotically stable, and

ii) there exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$, satisfying $\phi(0) = 0$, such that for all $w \in \ell_2^d$ and any $x_0 \in \mathbb{R}^n$,

$$\boxed{J := \|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \leqslant \phi(x_0). \qquad (5)}$$

The infimum value of $\gamma \in \mathbb{R}_{\succ 0}$ for which a τ-periodic ℓ_2-controller exists with ℓ_2-gain bound γ is called the infimal ℓ_2-gain of (1), and is denoted by γ_τ^*. Moreover, the problem of finding a τ-periodic ℓ_2-controller with a given ℓ_2-gain bound is called the ℓ_2-control problem. □

II-B Information state

In this section we recall the definition of the *information state* [24] for system (1). Then we determine the dynamics of the information state in Lemma 1 for a given initial condition. Finally, we mention how to solve the τ-periodic ℓ_2-control problem based on the information state.

Definition 3: (*Information state*, [23, Eq. 4.7]) Let an upper semicontinuous function $p_0 : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be given. Then for given $u_{0:t-1}$ and $y_{0:t-1}$, the information state $p_t(x_t) \in \mathbb{R} \cup \{-\infty\}$ at every $t \in \mathbb{N}$ for the system (1) is defined as

$$p_t(x_t) := \max_{w_{0:t-1} \in \ell^d_{2,0,t-1}} \max_{x_0 \in \mathbb{R}^n} \left\{ p_0(x_0) + \sum_{i=0}^{t-1} \|z_i\|^2 - \gamma^2 \|w_i\|^2; \right.$$

$$\left. \text{where for all } i \in \mathbb{N}_0^{t-1} : \quad \begin{aligned} x_{i+1} &= Ax_i + B_1 w_i + B_2 u_i, \\ z_i &= C_1 x_i + D_{12} u_i, \\ y_i &= C_2 x_i + D_{21} w_i, \end{aligned} \right\}. \tag{6}$$

□

The information state in Definition 3 is known as the largest value of a *cost to arrive* to a specific state x_t at a specific time step $t \in \mathbb{N}$ or as a *sufficient statistic* needed for the ℓ_2-controller design. In fact, it represents an upper bound of the cost function $(p_0(x_0) + \sum_{i=0}^{t-1} \|z_i\|^2 - \gamma^2 \|w_i\|^2)$ at every time step, given that all the control inputs $u_{0:t-1}$ and all the measured outputs $y_{0:t-1}$ are known up to the previous time step. Furthermore, it can be perceived as a bounded real lemma in state estimation, where $\mathcal{V}_t(x_t) := -p_t(x_t)$ for all $t \in \mathbb{N}_0$, then based on (6) we can easily show that

$$\mathcal{V}_0(x_0) \geqslant \mathcal{V}_t(x_t) + \textstyle\sum_{i=0}^{t-1} \|z_i\|^2 - \gamma^2 \|w_i\|^2.$$

In the next lemma, we determine the dynamics of the information state by solving (6) for a quadratic p_0. This becomes the basis of the τ-periodic ℓ_2-controller design in the next section. To simplify the notation, let us first define

$$Q := C_1^\mathsf{T} C_1, \quad R := D_{12}^\mathsf{T} D_{12}, \quad P := C_1^\mathsf{T} D_{12}, \quad L := B_1 D_{21}^\mathsf{T},$$
$$N := D_{21} D_{21}^\mathsf{T}, \quad \tilde{Q} := Q - PR^{-1}P^\mathsf{T}, \quad \hat{A} := A - LN^{-1}C_2,$$
$$\tilde{A} := A - B_2 R^{-1} P^\mathsf{T}, \quad \check{A} = A - LN^{-1}C_2 - B_2 R^{-1} P^\mathsf{T}.$$

We need the following assumptions in the sequel.

Assumption 1: It holds that
i) (A, B_2) is stabilizable, (C_2, A) is detectable,
ii) D_{21} is full row rank, $(\hat{A}, B_1 - LN^{-1}D_{21})$ is controllable,
iii) D_{12} is full column rank, and $(C_1 - D_{12}R^{-1}P^\mathsf{T}, \tilde{A})$ is observable. □

Assumption 1(i), D_{21} being full row rank and D_{12} being full column rank are standard assumptions in the literature (see, e.g., [22]). The second part part of Assumption 1(ii) is further discussed in Remark 1 and Assumption 1(iii) is an analogous condition to a condition provided in [22, Lem 3.5] to establish existence of solutions to infinite-horizon Riccati equations. These are technical assumptions, which we introduce as sufficient conditions to prove our results. These conditions might introduce some convervativeness to the class of system our results apply to.

Lemma 1: (*State-information for system* (1)) Suppose that Assumption 1 holds and for a given $\gamma \in \mathbb{R}_{\succ 0}$ and a positive definite matrix Y_0, let $p_0(x_0) := -\gamma^2 \|x_0\|^2_{Y_0^{-1}}$ for all $x_0 \in \mathbb{R}^n$.

i) The information state in Definition 3 is bounded at every $t \in \mathbb{N}$ and is given by

$$\boxed{p_t(x_t) = -\gamma^2 \|x_t - \bar{x}_t\|^2_{Y_t^{-1}} + Z_t, \qquad (7)}$$

under the condition that S_t be positive definite for all $t \in \mathbb{N}_0$, where

$$S_t := Y_t^{-1} + C_2^\mathsf{T} N^{-1} C_2 - \gamma^{-2} Q, \tag{8}$$

in which

$$Y_{t+1} = \hat{A} S_t^{-1} \hat{A}^\intercal + B_1 B_1^\intercal - L N^{-1} L^\intercal \tag{9}$$

for all $t \in \mathbb{N}_0$. Moreover, the vector $\bar{x}_t$ and the scalar Z_t are given for all $t \in \mathbb{N}_0$ by

$$\bar{x}_{t+1} = \bar{A}_t \bar{x}_t + \bar{B}_t \bar{u}_t + \bar{D}_t \bar{y}_t, \tag{10}$$

$$Z_{t+1} = Z_t + \|\bar{x}_t\|^2_{\bar{Q}_t} + \|\bar{u}_t\|^2_{\bar{R}_t} - \gamma^2 \|\bar{y}_t\|^2_{\bar{V}_t} \tag{11}$$

with $\bar{x}_0 = 0$ and $Z_0 = 0$, where

$$\begin{aligned} &\bar{A}_t = \tilde{A} \tilde{\Pi}_t^{-1}, && \tilde{\Pi}_t = I - \gamma^{-2} Y_t \tilde{Q}, \\ &\bar{B}_t = B_2 + \gamma^{-2} \hat{A} S_t^{-1} P, && \tilde{S}_t = S_t + \gamma^{-2}(Q - \tilde{Q}), \\ &\bar{D}_t = \tilde{A} \tilde{S}_t^{-1} C_2^\intercal N^{-1} + L \bar{V}_t, && \bar{Q}_t = \tilde{Q} \tilde{\Pi}_t^{-1}, \\ &\bar{R}_t = R + \gamma^{-2} P^\intercal S_t^{-1} P, \\ &\bar{V}_t = N^{-1}(I - C_2 \tilde{S}_t^{-1} C_2^\intercal N^{-1}), \end{aligned} \tag{12}$$

assuming $\tilde{\Pi}_t$ has no zero eigenvalue, with

$$\bar{u}_t = u_t - (\check{u}_t + \tilde{u}_t), \quad \bar{y}_t = y_t - \tilde{y}_t, \tag{13}$$

for

$$\check{u}_t := -\check{K}_t y_t, \quad \tilde{u}_t := -\tilde{K}_t \bar{x}_t, \quad \tilde{y}_t := \tilde{L}_t \bar{x}_t, \tag{14}$$

and

$$\begin{aligned} &\check{K}_t = R^{-1} P^\intercal \tilde{S}_t^{-1} C_2^\intercal N^{-1}, \ \tilde{K}_t = R^{-1} P^\intercal \tilde{S}_t^{-1} Y_t^{-1}, \\ &\tilde{L}_t = C_2 \tilde{\Pi}_t^{-1}. \end{aligned}$$

ii) If S_k determined based on (8) and (9) has a negative eigenvalue at a given $k \in \mathbb{N}_0$, then $p_t(x_t)$ determined based on (6) is unbounded, i.e. $+\infty$, for all $t \in \mathbb{N}_{\geqslant k+1}$. □

Remark 1: Based on (7), the calculation of the information state at every time-step requires Y_t to be invertible. It is worth mentioning that at every $t \in \mathbb{N}$, $S_{t-1} \succ 0$ and the controllability of $(\hat{A}, B_1 - L N^{-1} D_{21})$, by Assumption 1(ii), are sufficient conditions to guarantee that Y_t determined based on (9) is positive definite (see the proof of Lemma 1). □

Remark 2: We do not cover the case in which S_k has one or multiple zero eigenvalues at a given $k \in \mathbb{N}_0$, see Remark 3 below, for a more detailed discussions. □

Note that based on (10) and (11), the values of $\bar{x}_t$ and Z_t can always be determined if the information set $\{y_i, u_i | i \in \mathbb{N}_0^{t-1}\}$ is available at every $t \in \mathbb{N}$. In order to determine a τ-periodic ℓ_2-controller, we first consider the following minimax optimization problem over a finite-horizon T:

$$\begin{aligned} \hat{J}^* = &\max\nolimits_{\{y_t := \mathcal{T}_t(\mathcal{E}_t)\}_{t \in \mathbb{N}_0^{T-1}}} \\ &\min\nolimits_{\{u_t = \mathcal{R}_t^{\pi_\tau}(\mathcal{F}_t^{\pi_\tau})\}_{t \in \mathbb{N}_0^{T-1}}} \hat{J}(y_{0:T-1}, u_{0:T-1}), \end{aligned} \tag{15}$$

where

$$\hat{J}(y_{0:T-1}, u_{0:T-1}) := \max_{x_T \in \mathbb{R}^n} [p_T(x_T) + q(x_T)],$$

for some mapping $\mathcal{T}_t : \mathcal{E}_t \to \mathbb{R}^o$, $t \in \mathbb{N}_0$,

$$\mathcal{E}_t := \{y_i | i \in \mathbb{N}_0^{t-1}\}, \tag{16}$$

at every $t \in \mathbb{N}_0$, and

$$\begin{aligned} q(x_T) := &\min\nolimits_{u_k := \mathcal{M}(x_k)_{k \in \mathbb{N}_{\geqslant T}}} \\ &\max\nolimits_{w_k := \mathcal{N}(x_k)_{k \in \mathbb{N}_{\geqslant T}}} \left(\textstyle\sum_{k=T}^{\infty} \|z_k\|^2 - \gamma^2 \|w_k\|^2 \right) \end{aligned} \tag{17}$$

is the value function of the full-state feedback zero-sum dynamic game in which $\mathcal{M}$ and $\mathcal{N}$ are appropriate mappings. Moreover, $p_T(x_T)$ follows (7) determined in Lemma 1. Note that $\hat{J}(y_{0:T-1}, u_{0:T-1})$ is independent of the disturbance inputs since both $p_T(x_T)$ and $q_T(x_T)$ are independent from the disturbance inputs. The control policy determined by the solution of (15) can be shown to solve a output feedback ℓ_2-control problem over the finite-horizon [25]. After we determine such a controller by solving (15), we will find the conditions under which the determined policy can be the solution of the (infinite-horizon) τ-periodic ℓ_2-controller as introduced in Definition 2. This is carried out in the following subsections.

II-C 1*-Periodic* ℓ_2*-controller*

The next lemma gives sufficient conditions for the existence of a 1-periodic ℓ_2-controller for a given $\gamma \in \mathbb{R}_{\succ 0}$, and provides a 1-periodic ℓ_2-controller with ℓ_2-gain bound γ. The conditions are given in terms of positive-definiteness of given matrices. The lemma also states a converse result, showing that these conditions are strict. Namely if one of these matrices has a negative eigenvalue then no 1-periodic ℓ_2-controller for the given $\gamma \in \mathbb{R}_{\succ 0}$ exists. Recall that γ_1^* is called the infimal ℓ_2-gain of (1) based on Definition 2. The proof can be found in the appendix and relies on the solution to (15).

Lemma 2: (1*-Periodic* ℓ_2*-controller*) Let Assumption 1 hold and consider a given $\gamma \in \mathbb{R}_{\succ 0}$. Suppose the following conditions hold

$$\boxed{\begin{aligned} &\Delta_1 := \gamma^2 I - Y X \succ 0, \quad \Delta_2 := S \succ 0, \\ &\Delta_3 := \gamma^2 I - B_1^\top X B_1 \succ 0, \end{aligned} \qquad (18)}$$

where X and Y are the positive definite solutions of

$$X = \tilde{A}^\intercal X H^{-1} \tilde{A} + \tilde{Q}, \tag{19}$$

$$Y = \hat{A} S^{-1} \hat{A}^\intercal + B_1 B_1^\intercal - L N^{-1} L^\intercal, \tag{20}$$

for

$$\begin{aligned} &H = I + (B_2 R^{-1} B_2^\intercal - \gamma^{-2} B_1 B_1^\intercal) X, \\ &S = Y^{-1} + C_2^\intercal N^{-1} C_2 - \gamma^{-2} Q. \end{aligned} \tag{21}$$

Then, $\gamma \succ \gamma_1^*$ and

i) the control policy

$$\boxed{u_t^* = -R^{-1} P^\top \bar{x}_t^u - K(\check{A} \bar{x}_t^u + L N^{-1} y_t), \quad (22)}$$

is a 1-periodic ℓ_2-controller with ℓ_2-gain bound γ based on Definition 2, where

$$K := R^{-1}(B_2^\intercal + \gamma^{-2} P^\intercal \tilde{S}^{-1} \check{A}^\intercal) G, \tag{23}$$

for

$$\begin{aligned} &\tilde{S} = Y^{-1} + C_2^\intercal N^{-1} C_2 - \gamma^{-2} \tilde{Q}, \\ &G = (H X^{-1} + \gamma^{-2} L N^{-1} L^\intercal - \gamma^{-2} \check{A} \tilde{S}^{-1} \check{A}^\intercal)^{-1}, \end{aligned} \tag{24}$$

and

$$\bar{x}_t^u := \tilde{S}^{-1}(Y^{-1}\bar{x}_t + C_2^\intercal N^{-1} y_t), \tag{25}$$

with the state estimator

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t + \bar{B}(u_t - \check{u}_t - \tilde{u}_t) + \bar{D}(y_t - \tilde{y}_t), \tag{26}$$

for all $t \in \mathbb{N}_0$, with $\bar{x}_0 = 0$,

$$\begin{aligned} &\bar{A} = \tilde{A}\tilde{\Pi}^{-1}, && \bar{B} = B_2 + \gamma^{-2}\hat{A}S^{-1}P, \\ &\tilde{\Pi} = I - \gamma^{-2}Y\tilde{Q}, && \bar{D} = \tilde{A}\tilde{S}^{-1}C_2^\intercal N^{-1} + L\bar{V}, \\ &\bar{V} = N^{-1}(I - C_2\tilde{S}^{-1}C_2^\intercal N^{-1}), \end{aligned}$$

and

$$\check{u}_t = -\check{K}y_t, \qquad \tilde{u}_t = -\tilde{K}\bar{x}_t, \qquad \tilde{y}_t = \tilde{L}\bar{x}_t, \tag{27}$$

for

$$\begin{aligned} &\check{K} = R^{-1}P^\intercal\tilde{S}^{-1}C_2^\intercal N^{-1}, \quad \tilde{K} = R^{-1}P^\intercal\tilde{S}^{-1}Y^{-1}, \\ &\tilde{L} = C_2\tilde{\Pi}^{-1}. \end{aligned}$$

ii) the performance index J in (5) is upper bounded by

$$\boxed{J \leqslant \gamma^2\|x_0\|^2_{Y^{-1}} + \sum_{t=0}^{\infty}\|u_t - u_t^*\|^2_{\Phi} - \|y_t - y_t^*\|^2_{\Psi},} \tag{28}$$

for

$$y_t^* := (C_2 + \gamma^{-2}L^\intercal XH^{-1}\tilde{A})\Delta_1^{-1}\bar{x}_t, \tag{29}$$

and having $\bar{R} := R + \gamma^{-2}P^\intercal S^{-1}P$, then

$$\Phi := \bar{R} + \bar{B}^\intercal X\Delta_1^{-1}\bar{B}, \;\; \Psi := \gamma^2\bar{V} - \bar{D}^\intercal G\bar{D}. \tag{30}$$

Moreover, if for a given $\gamma \in \mathbb{R}_{\succ 0}$, one of the matrices Δ_1, Δ_2 or Δ_3 in (18) has a negative eigenvalue, then $\gamma \prec \gamma_1^*$. □

Remark 3: We do not cover the case where for a given $\gamma \in \mathbb{R}_{\succ 0}$, one of the matrices Δ_1, Δ_2 or Δ_3 has one or multiple zero eigenvalues. Note that typically, the set of $\gamma \in \mathbb{R}$ for which this is the case, has zero measure. Therefore, from a practical point of view this is not a severe limitation of our work. It is worth mentioning this is the only reason we cannot call the conditions in (18) necessary and sufficient for the existence of 1-periodic ℓ_2-controller. □

Remark 4: Under Assumption 1, the solutions of the stationary Riccati equations (19) and (20) for a given $\gamma \in \mathbb{R}_{\succ 0}$ should be positive definite so that the control policy (22) solves the 1-periodic ℓ_2-control problem. In fact, X is needed to be positive definite even for the state-feedback ℓ_2-control problem, see, e.g., [22]. Moreover, based on Remark 1, under the condition that $\Delta_2 \succ 0$, Y is positive definite. □

II-D τ-Periodic ℓ_2-controller for a $\tau \in \mathbb{N}$

This section focuses on a multi-rate periodic ℓ_2-control problem in which the controller has access to the measured outputs only every τ time-steps, but then also gets all intermediate/previous values. Recall that γ_τ^* is called the infimal ℓ_2-gain of (1) based on Definition 2. Note that the output policy that is determined based on (15) depends on all the measured outputs from the initial up to the previous time-step, i.e., up to $t-1$. We introduce another time variable $\kappa \in \mathbb{N}_0$, where $\kappa = \lfloor \frac{t}{\tau} \rfloor$ and define the augmented control inputs and measured outputs at every κ as $U_\kappa = [u_{\kappa\tau}^\intercal \dots u_{\kappa\tau+\tau-1}^\intercal]^\intercal$ and $\bar{Y}_\kappa = [y_{\kappa\tau}^\intercal \dots y_{\kappa\tau+\tau-1}^\intercal]^\intercal$. Lemma 3 presents a τ-periodic ℓ_2-controller for a general $\tau \in \mathbb{N}$ based on the solution of (15).

Lemma 3: (*τ-Periodic ℓ_2-controller for $\tau \in \mathbb{N}$*) Let Assumption 1 hold and consider a given $\gamma \in \mathbb{R}_{\succ 0}$. Suppose the following inequalities hold

$$\boxed{\begin{aligned} &\Delta_1 \succ 0, \quad \Delta_2 \succ 0, \quad \Delta_3 \succ 0, \quad \text{and } \forall h \in \mathbb{N}_2^\tau, \\ &\Delta_{2+h} := \gamma^2 I - (A\Pi^{-1}YA^\top + B_1B_1^\top)\Theta_h \succ 0, \end{aligned}} \tag{31}$$

for $\Pi = I - \gamma^{-2}YQ$, where $\Delta_{1,2,3}$ are given in (18), X, Y are the positive definite solutions of (19), (20), respectively, and

$$\Theta_h := \begin{cases} A^\intercal\Theta_{h+1}V_h^{-1}A + Q, & \text{if } h \in \mathbb{N}_1^{\tau-1} \\ X, & \text{if } h = \tau, \end{cases} \tag{32}$$

where $V_h := I - \gamma^{-2}B_1B_1^\intercal\Theta_{h+1}, \quad \forall h \in \mathbb{N}_1^{\tau-1}$. Then, $\gamma \succ \gamma_\tau^*$ and

i) the control policy

$$\boxed{\begin{aligned} &u_t^* = \\ &\begin{cases} -R^{-1}P^\top\bar{x}_t^u - K(\check{A}\bar{x}_t^u + LN^{-1}y_t), & \text{if } \pi_t^\tau = 1, \\ -R^{-1}(B_2^\top XH^{-1}\tilde{A} + P^\top)\hat{x}_{t|t}, & \text{otherwise,} \end{cases} \end{aligned}} \tag{33}$$

is a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ based on Definition 2, where π_t^τ, K, $\bar{x}_t^u$ and $\bar{x}_t$ are given in (2), (23), (25), and (26), respectively. Moreover,

$$\begin{aligned} &\hat{x}_{t+1|t} := H^{-1}\tilde{A}\hat{x}_{t|t}, \qquad \forall t \in \mathbb{N} \\ &\hat{x}_{t|t} := \begin{cases} X^{-1}G(\check{A}\bar{x}_{t-1}^u + LN^{-1}y_{t-1}), & \text{if } \pi_{t-1}^\tau = 1, \\ \hat{x}_{t|t-1}, & \text{otherwise,} \end{cases} \end{aligned} \tag{34}$$

where $\bar{x}_t^u$ and G follow (25) and (24), respectively.

ii) the performance index J in (5) is upper bounded by

$$\boxed{J \leqslant \gamma^2\|x_0\|^2_{Y^{-1}} + \sum_{\kappa=0}^{\infty}\|U_\kappa - U_\kappa^*\|^2_{\bar{\Phi}_\tau} - \|\bar{Y}_\kappa - \bar{Y}_\kappa^*\|^2_{\bar{\Psi}_\tau},} \tag{35}$$

where for every $\kappa \in \mathbb{N}_0$,

$$U_\kappa^* := [u_{\kappa\tau}^{*\intercal} \dots u_{\kappa\tau+\tau-1}^{*\intercal}]^\intercal, \quad \bar{Y}_\kappa^* := [y_{\kappa\tau}^{*\intercal} \dots y_{\kappa\tau+\tau-1}^{*\intercal}]^\intercal, \tag{36}$$

in which u_t^* follows (33) for all $t \in \mathbb{N}_0$,

$$y_{\kappa\tau}^* := (C_2 + \gamma^{-2}L^\intercal XH^{-1}\tilde{A})\Delta_1^{-1}\bar{x}_{\kappa\tau}, \tag{37}$$

and for all $i \in \mathbb{N}_1^{\tau-1}$,

$$y_{\kappa\tau+i}^* := \bar{y}_{\kappa\tau+i}^* + C_2\Pi^{-1}(\bar{x}_{\kappa\tau+i} + \gamma^{-2}YPu_{\kappa\tau+i}), \tag{38}$$

where

$$\begin{aligned} \bar{y}_{\kappa\tau+i}^* := \gamma^{-2}&(C_2\Pi^{-1}YA^\intercal + L^\intercal) \\ &(\Theta_{i+1}V_i^{-1}A\Omega_i\bar{x}_{\kappa\tau+i} + \Xi_{i+1}\Lambda_i\hat{U}_{\kappa\tau+i}), \end{aligned}$$

in which $\hat{U}_{\kappa\tau+i} := [u_{\kappa\tau+i}^{\mathsf{T}} \dots u_{\kappa\tau+\tau-1}^{\mathsf{T}}]^{\mathsf{T}}$,

$$\Lambda_i := \begin{cases} \hat{B}, & \text{if } i = \tau - 1, \\ [\hat{B} \quad \Theta_{i+1}^{-1}\hat{N}_{i+1}^{\mathsf{T}}], & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{N}_1^{\tau}$, where $\hat{B} := B_2 + \gamma^{-2}A\Pi^{-1}YP$,

$$\begin{aligned} \Xi_i &:= (V_{i-1}\Theta_i^{-1} - \gamma^{-2}A\Pi^{-1}YA^{\mathsf{T}})^{-1}, \\ \Omega_i &:= (I - \gamma^{-2}Y\Theta_i)^{-1}, \end{aligned}$$

and having $\hat{N}_{\tau-1} := B_2^{\mathsf{T}}\Theta_\tau V_{\tau-1}^{-1}A + P^{\mathsf{T}}$, then

$$\hat{N}_i := \begin{bmatrix} B_2^{\mathsf{T}}\Theta_{i+1}V_i^{-1}A + P^{\mathsf{T}} \\ \hat{N}_{i+1}V_i^{-1}A \end{bmatrix}, \quad \forall i \in \mathbb{N}_1^{\tau-2}.$$

Moreover, $\bar{\Psi}_\tau := \text{diag}(\Psi_0, \dots, \Psi_{\tau-1})$ such that

$$\Psi_i := \begin{cases} \gamma^2\bar{V} - \bar{D}^{\mathsf{T}}G\bar{D}, & \text{if } i = 0, \\ \gamma^2\hat{V} - \hat{D}^{\mathsf{T}}\Theta_{i+1}\Omega_{i+1}\hat{D}, & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{N}_1^{\tau-1}$, G in (24) and

$$\begin{aligned} \hat{V} &:= N^{-1}(I - C_2S^{-1}C_2^{\mathsf{T}}N^{-1}), \\ \hat{D} &:= AS^{-1}C_2^{\mathsf{T}}N^{-1} + L\hat{V}. \end{aligned}$$

Furthermore,

$$\bar{\Phi}_\tau := \begin{bmatrix} \bar{R} + \bar{B}^{\mathsf{T}}\Theta_1\Omega_1\bar{B} & \bar{B}^{\mathsf{T}}\Omega_1^{\mathsf{T}}\hat{N}_1^{\mathsf{T}} \\ \hat{N}_1\Omega_1\bar{B} & \bar{\Upsilon}_1 \end{bmatrix},$$

where for all $i \in \mathbb{N}_1^{\tau-2}$,

$$\begin{aligned} \bar{\Upsilon}_i := \Lambda_i^{\mathsf{T}}\Xi_{i+1}\Lambda_i + \text{diag}(&R + \gamma^{-2}P^{\mathsf{T}}\Pi^{-1}YP, \\ &\bar{\Upsilon}_{i+1} - \hat{N}_{i+1}\Theta_{i+1}^{-1}\Omega_{i+1}^{\mathsf{T}}\hat{N}_{i+1}^{\mathsf{T}}), \end{aligned}$$

such that $\bar{\Upsilon}_{\tau-1} := R + \hat{B}^{\mathsf{T}}\Xi_\tau\hat{B} + \gamma^{-2}P^{\mathsf{T}}\Pi^{-1}YP$.

Moreover, if for a given $\gamma \in \mathbb{R}_{\succ 0}$, one of the matrices Δ_i, for $i \in \mathbb{N}_1^{2+\tau}$ in (31) has a negative eigenvalue, then $\gamma \prec \gamma_\tau^*$.
□

Similar to Remark 3, we do not investigate the situation in which for a specific $\gamma \in \mathbb{R}_{\succ 0}$, one of the matrices Δ_i, for $i \in \mathbb{N}_1^{2+\tau}$, has one or multiple zero eigenvalues.

Based on (33) and (34), the controller just needs to receive $\{\bar{x}_t, y_t\}$ at every $t = \tau\kappa$ in order to calculate the control input at every $t \in \mathbb{N}_0$. Therefore, for the proposed τ-periodic ℓ_2-controller, the measured outputs after the last transmission time do not need to be transmitted to the controller.

II-E Networked control system for event-triggered communication

The NCS we are interested in the ETC setting has the same plant G as given in (1). However, data transmission from the sensors to the controller are now triggered by an output-dependent mechanism, which is called an event-triggered transmission policy. This event-triggered transmission policy can be formalized as $\mu = \{\mu_t\}_{t \in \mathbb{N}_0}$, where

$$\delta_t = \mu_t(\mathcal{H}_t) \in \{0, 1\} \tag{39}$$

with

$$\mathcal{H}_t := \{y_i | i \in \mathbb{N}_0^t\} \cup \{\delta_i | i \in \mathbb{N}_0^{t-1}\} \tag{40}$$

is the information set available for the scheduler at every $t \in \mathbb{N}_0$ to decide whether to transmit ($\delta_t = 1$) or not to transmit ($\delta_t = 0$) at time t. Then, any appropriate control policy is formalized as $\mathcal{R}^\mu = \{\mathcal{R}_t^\mu\}_{t \in \mathbb{N}_0}$, where

$$u_t = \mathcal{R}_t^\mu(\mathcal{F}_t^\mu) \tag{41}$$

for a mapping $\mathcal{R}_t^\mu : \mathcal{F}_t^\mu \to \mathbb{R}^m$. The information set $\mathcal{F}_t^\mu$ available for the controller contains all the measured outputs up to the last triggering time. However, based on the periodic time-triggered control policies in (22) and (33), $\bar{x}_t$ at every $t \in \mathbb{N}$ is a sufficient representative of all the measured outputs up to the time step $t - 1$ needed by the controller. This indicates that at every triggering time step $i \in \mathbb{N}_0$, instead of $\{y_{\mathsf{L}_{i-1}+1}, \dots, y_i\}$, the scheduler only needs to transmit $\{\bar{x}_i, y_i\}$ to the controller. Accordingly,

$$\mathcal{F}_t^\mu := \{\bar{x}_i, y_i | i \in \mathbb{N}_0^t \wedge \mu_i(\mathcal{H}_i) = 1\}, \tag{42}$$

where $\bar{x}_i$ is determined based on (26) at every $i \in \mathbb{N}_0$ in the scheduler. We call a combined event-triggered scheduler and controller $\eta = (\mu, \mathcal{R}^\mu)$ an ETC. Furthermore, we introduce the average transmission rate associated with an ETC policy η as $\bar{f}_\eta(x_0, w) = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mu_t(\mathcal{H}_t)$ and the average inter-transmission time as $\bar{\omega}_\eta(x_0, w) = 1/\bar{f}_\eta(x_0, w)$. Note that this depends on the initial state x_0 and the disturbance realization w. Next, we define the concept of an event-triggered ℓ_2-controller.

Definition 4: (*Event-triggered ℓ_2-controller*) Given $\gamma \in \mathbb{R}_{\succ 0}$, an ETC $\eta = (\mu, \mathcal{R}^\mu)$ for the system (1) is referred to as an event-triggered ℓ_2-controller with ℓ_2-gain bound γ, if

i) the closed-loop control system is globally asymptotically stable, and
ii) there exists a function $\phi : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ satisfying $\phi(0) = 0$, such that for all $w \in \ell_2^d$, any $x_0 \in \mathbb{R}^n$, (5) holds. □

II-F Problem statement

We are interested in decreasing the communication frequency between the sensors and the controller by employing the idea of ETC, while at the same time preserving the same ℓ_2-gain bound for the control loop as the designed τ-periodic ℓ_2-controllers in (22) and (33). This concept is introduced in the following definition as an ℓ_2-consistent ETC.

Definition 5: (ℓ_2-consistent event-triggered controller) For any given $\tau \in \mathbb{N}$ and any ℓ_2-gain bound $\gamma \in \mathbb{R}_{\succ 0}$ for the system (1), where the τ-periodic ℓ_2-controller (33) (or equivalently (22), in case $\tau = 1$) exists, an event-triggered ℓ_2-controller $\eta = (\mu, \mathcal{R}^\mu)$ is said to be ℓ_2-consistent with ℓ_2-gain bound γ, if

i) η has ℓ_2-gain bound γ, and
ii) in comparison to the τ-periodic ℓ_2-controller (33) (or equivalently (22), in case $\tau = 1$) with ℓ_2-gain bound γ, the average transmission rate of η is lower than or at most equal to $\frac{1}{\tau}$, i.e., $\bar{f}_\eta(x_0, w) \leqslant \frac{1}{\tau}$ for all x_0 and all $w \in \ell_2^d$. □

Based on Definition 5, the goal of this work is to propose an ℓ_2-consistent ETC for the NCS depicted in Fig. 1.

III MAIN RESULTS: ℓ_2-CONSISTENT EVENT-TRIGGERED CONTROLLER

For simplicity, in Section III-A first we propose an ℓ_2-consistent ETC based on Definition 5 for $\tau = 1$. Then, in Section III-B, we consider the general case.

III-A *Special case $\tau = 1$*

An ETC consists of two main parts being the controller and the scheduler. Inspired by the 1-periodic ℓ_2-controller in (22), we propose the controller of our ℓ_2-consistent ETC as

$$\boxed{\mathcal{R}_t^{\mu}(\mathcal{F}_t^{\mu}) = -R^{-1}P^{\top}\tilde{x}^u_{t|t} - K(\check{A}\tilde{x}^u_{t|t} + LN^{-1}\tilde{y}_t), \qquad (43)}$$

where K is given as in (23),

$$\tilde{x}^u_{t|t} = \tilde{S}^{-1}(Y^{-1}\tilde{x}_{t|t} + C_2^{\intercal}N^{-1}\tilde{y}_t), \qquad (44)$$

and $\tilde{y}_t$, $\tilde{x}_{t|t}$ are the estimation of y_t and $\bar{x}_t$ in the controller, respectively, at every $t \in \mathbb{N}_0$. When the scheduler triggers a transmission, i.e., $\delta_t = \mu_t(\mathcal{H}_t) = 1$, $\{\bar{x}_t, y_t\}$ is communicated to the controller and, therefore, $\tilde{y}_t = y_t$, $\tilde{x}_{t|t} = \bar{x}_t$. However, in case of no data transmission, we use the hold policy for the estimation of y_t in the controller. Therefore, having $\tilde{y}_{-1} = 0$, for all $t \in \mathbb{N}_0$,

$$\tilde{y}_t = \begin{cases} y_t, & \text{if } \mu_t(\mathcal{H}_t) = 1 \\ \tilde{y}_{t-1}, & \text{otherwise.} \end{cases} \qquad (45)$$

The main reason for not employing a model-based estimation for y_t is due to the fact that within the context of the ℓ_2-control problem, the characteristics of the disturbance input w is usually unknown, where it can take any bounded value. Now for $\bar{x}_{t|t}$, we introduce the following model-based estimator,

$$\begin{aligned} \tilde{x}_{t+1|t} &= \bar{A}\tilde{x}_{t|t} + \bar{B}(u_t + \check{K}\tilde{y}_t + \tilde{K}\tilde{x}_{t|t}) + \bar{D}(\tilde{y}_t - \tilde{L}\tilde{x}_{t|t}), \\ \tilde{x}_{t|t} &= \begin{cases} \bar{x}_t, & \text{if } \mu_t(\mathcal{H}_t) = 1 \\ \tilde{x}_{t|t-1}, & \text{otherwise,} \end{cases} \end{aligned} \qquad (46)$$

where $\tilde{x}_{0|0} = 0$ and $\tilde{y}_t$ follows (45) for all $t \in \mathbb{N}_0$. Next, we define the event-triggered transmission policy μ_t as

$$\boxed{\mu_t(\mathcal{H}_t) := \begin{cases} 1, & \text{if } \beta_{t+1|t} \succ 0 \\ 0, & \text{otherwise,} \end{cases} \qquad (47)}$$

for all $t \in \mathbb{N}_0$, where

$$\begin{aligned} \beta_{t+1|t} &:= \beta_{t|t} - \|y_t - y_t^*\|^2_{\Psi} + \|u_t^p - u_t^*\|^2_{\Phi}, \\ \beta_{t+1|t+1} &:= \begin{cases} \beta_{t|t} - \|y_t - y_t^*\|^2_{\Psi}, & \text{if } \mu_t(\mathcal{H}_t) = 1 \\ \beta_{t+1|t}, & \text{otherwise,} \end{cases} \end{aligned}$$

for all $t \in \mathbb{N}_0$, $\beta_{0|0} := 0$ and

$$u_t^p := -R^{-1}P^{\intercal}\tilde{x}^u_{t|t-1} - K(\check{A}\tilde{x}^u_{t|t-1} + LN^{-1}\tilde{y}_{t-1}),$$

where

$$\tilde{x}^u_{t|t-1} = \tilde{S}^{-1}(Y^{-1}\tilde{x}_{t|t-1} + C_2^{\intercal}N^{-1}\tilde{y}_{t-1}). \qquad (48)$$

Moreover, for all $t \in \mathbb{N}_0$, $\tilde{y}_{t-1}$, $\tilde{x}_{t|t-1}$, u_t^* and y_t^* follow (45), (46), (22) and (29), respectively. The next theorem states the ℓ_2-consistency of the proposed ETC, i.e. (43) and (47).

Theorem 1: (*ℓ_2-consistency of the ETC for $\tau = 1$*) Let Assumption 1 hold. Then for system (1), the ETC with the scheduling policy (47) and the control policy (43), where $\tilde{y}_t$, $\tilde{x}_{t|t}$ follow (45) and (46), respectively, is ℓ_2-consistent in the sense of Definition 5. ☐

The main idea behind the proposed ℓ_2-consistent ETC comes directly from the upper bound of J in (28). As the measured outputs deviate from y_t^* at some $t \in \mathbb{N}_0$, it results in some negative terms, which provides the opportunity to skip data transmissions at some times. In principle, the proposed ETC balances the negative terms of cost upper bound arising when $y_t \neq y_t^*$, with the positive terms of the cost upper bound, when $u_t \neq u_t^*$ due to skipped data transmissions, such that still $J \leqslant \gamma^2\|x_0\|^2_{Y^{-1}}$ holds.

III-B *General case $\tau \in \mathbb{N}$*

Similar to the previous section, we first introduce the control input of our desired ETC. Based on the periodic ℓ_2-controller (33), where the state estimation $\bar{x}_t$ and the current measured output y_t are transmitted to the controller every $\tau \in \mathbb{N}$ time-steps, the control input of the ETC is proposed as

$$\boxed{\begin{aligned} &\mathcal{R}_t^{\mu}(\mathcal{F}_t^{\mu}) := \\ &\begin{cases} -R^{-1}P^{\top}\tilde{x}^u_{t|t} - K(\check{A}\tilde{x}^u_{t|t} + LN^{-1}\tilde{y}_t), & \text{if } \pi_t^{\tau} = 1, \\ -R^{-1}(B_2^{\top}XH^{-1}\tilde{A} + P^{\top})\check{x}_{t|t}, & \text{otherwise,} \end{cases} \qquad (49) \end{aligned}}$$

at every $t \in \mathbb{N}_0$, where π_t^{τ}, $\tilde{x}^u_{t|t}$, $\tilde{y}_t$, $\tilde{x}_{t|t}$ follow (2), (44), (45) and (46), respectively, and for every $t \in \mathbb{N}$,

$$\begin{aligned} \check{x}_{t+1|t} &= H^{-1}\tilde{A}\check{x}_{t|t}, \\ \check{x}_{t|t} &= \begin{cases} X^{-1}G(\check{A}\tilde{x}^u_{t-1|t-1} + LN^{-1}\tilde{y}_{t-1}), & \text{if } \pi_{t-1}^{\tau} = 1, \\ \check{x}_{t|t-1}, & \text{otherwise.} \end{cases} \end{aligned} \qquad (50)$$

Then, the event-triggered policy is proposed to be

$$\boxed{\mu_t(\mathcal{H}_t) := \begin{cases} 1, & \text{if } \pi_t^{\tau} = 1 \text{ and } \beta_{(\kappa+1)\tau|\kappa\tau} \succ 0 \\ 0, & \text{otherwise,} \end{cases} \qquad (51)}$$

for $t \in \mathbb{N}_0$, where considering $\tilde{\Psi}_{\tau} = \operatorname{diag}(\Psi_1, \ldots, \Psi_{\tau-1}, \Psi_0)$,

$$U_{\kappa}^p = \begin{bmatrix} u^p_{\kappa\tau} \\ \cdots \\ u^p_{\kappa\tau+\tau-1} \end{bmatrix}, \quad \tilde{Y}_{\kappa} = \begin{bmatrix} y_{\kappa\tau-\tau+1} \\ \cdots \\ y_{\kappa\tau} \end{bmatrix}, \quad \tilde{Y}^*_{\kappa} = \begin{bmatrix} y^*_{\kappa\tau-\tau+1} \\ \cdots \\ y^*_{\kappa\tau} \end{bmatrix},$$

for all $\kappa \in \mathbb{N}$, where

$$\begin{aligned} &u_t^p := \\ &\begin{cases} -R^{-1}P^{\intercal}\tilde{x}^u_{t|t-1} - K(\check{A}\tilde{x}^u_{t|t-1} + LN^{-1}\tilde{y}_{t-1}), & \text{if } \pi_t^{\tau} = 1, \\ -R^{-1}(B_2^{\intercal}XH^{-1}\tilde{A} + P^{\intercal})\check{x}_{t|t-1}, & \text{otherwise,} \end{cases} \end{aligned} \qquad (52)$$

where $\tilde{x}^u_{t|t-1}$ follows (48),

$$\check{\check{x}}_{t+1|t} = \begin{cases} X^{-1}G(\check{A}\tilde{x}^u_{t|t-1} + LN^{-1}\tilde{y}_{t-1}), & \text{if } \pi^\tau_t = 1, \\ H^{-1}\tilde{A}\check{\check{x}}_{t|t-1}, & \text{otherwise,} \end{cases} \tag{53}$$

$\beta_{1|1} := -\|y_0\|^2_{\Psi_0}$ and for all $\kappa \in \mathbb{N}_0$,

$$\beta_{(\kappa+1)\tau|\kappa\tau} := \beta_{\kappa\tau|\kappa\tau} - \|\tilde{Y}_\kappa - \tilde{Y}^*_\kappa\|^2_{\tilde{\Psi}_\tau} + \|U^p_\kappa - U^*_\kappa\|^2_{\bar{\Phi}_\tau},$$

$$\beta_{(\kappa+1)\tau|(\kappa+1)\tau} := \begin{cases} \hat{\beta}_{(\kappa+1)\tau|\kappa\tau}, & \text{if } \mu_{\kappa\tau}(\mathcal{H}_{\kappa\tau}) = 1 \\ \beta_{(\kappa+1)\tau|\kappa\tau}, & \text{otherwise,} \end{cases}$$

in which $\hat{\beta}_{(\kappa+1)\tau|\kappa\tau} := \beta_{\kappa\tau|\kappa\tau} - \|\tilde{Y}_\kappa - \tilde{Y}^*_\kappa\|^2_{\tilde{\Psi}_\tau}$ and $\tilde{x}_{t|t-1}$, U^*_κ, $y^*_{\kappa\tau-\tau+1:\kappa\tau}$ follow (46), (36) and (37), (38), respectively.

Theorem 2: (ℓ_2-consistency of the ETC for $\tau \in \mathbb{N}$) Let Assumption 1 hold. Then for system (1), the ETC with the event-triggered scheduling policy (51) and the control input (49), where $\tilde{y}_t$, $\tilde{x}_{t|t}$ follow (45) and (46), respectively, is ℓ_2-consistent in the sense of Definition 5. □

Hence, also in the general case ℓ_2-consistent ETC can be designed systematically.

IV NUMERICAL EXAMPLE

Consider system (1) with

$$A = \begin{bmatrix} 0.8 & -0.5 \\ -0.25 & 1.1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix}, \; B_2 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0.2 \\ -0.3 & 0.5 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.5 & -0.1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0.5 & 1 \end{bmatrix},$$

where clearly Assumption 1 holds. Moreover, we take $w_t = e^{-\frac{t}{200}} \left[\sin(\frac{t}{15}) \quad \cos(\frac{t}{20})\right]^\mathsf{T}$, $t \in \mathbb{N}_0$, as the unknown disturbance input of the system and $x_0 = 0$. We denote the infimal ℓ_2-gain bound for which the τ-periodic ℓ_2-controller (33) exists by γ^*_τ as before. For the given system, these values are determined by a bisection method to be $\gamma^*_1 = 2.9550$, $\gamma^*_2 = 3.8055$, $\gamma^*_3 = 5.1140$, $\gamma^*_4 = 6.8489$, see Fig. 3. We will show now that the proposed ℓ_2-consistent ETC (based on Theorems 1 and 2) can result in significant improvements both in terms of the ℓ_2-gain bound and the communication rate.

To do so, consider Fig. 2 in which the state trajectories of two different closed-loop scenarios over a 250 time step window are shown. In the first, we display the PTC scenario, in which the scheduler is transmitting periodically with $\tau = 4$ and the controller follows (33) for $\gamma = 6.8510 = \gamma^*_4 + \epsilon$ close to the infimal value γ^*_4 as $\epsilon = 0.0021$ is a small real number. The second scenario is an ℓ_2-consistent ETC designed for $(\tau, \gamma) = (1, 3.0504)$ based on (43) and (47) (note $\gamma \succ \gamma^*_1$). The average transmission time period of this event-triggered scenario is $\bar{\omega}_\eta(x_0, w) = 4.032 \approx \tau = 4$ of the PTC. The comparison of these two scenarios indicates that it is possible to guarantee lower ℓ_2-gain bounds for the system (1) by employing the proposed ℓ_2-consistent ETC in comparison to the periodic ℓ_2-controller in Lemmas 2 and 3, while they both have almost the same average transmission rate from the scheduler to the controller for the corresponding disturbance input. The better disturbance attenuation of the ℓ_2-consistent ETC can be easily seen in Fig. 2 and is guaranteed by our theory.

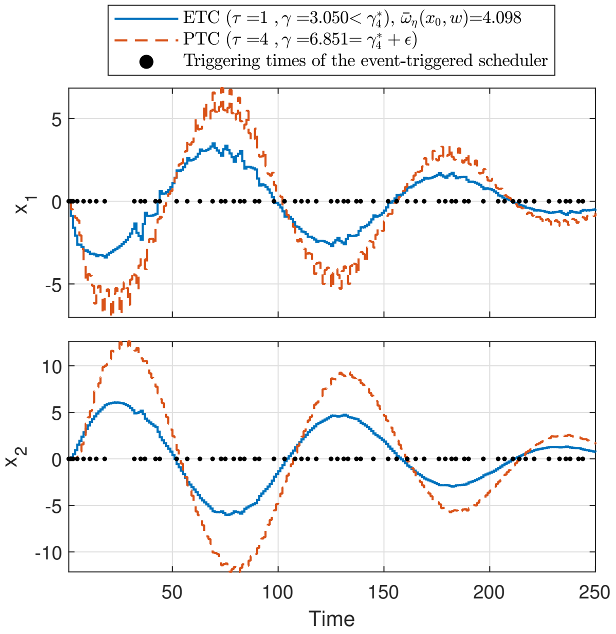

Fig. 2: Illustration of the higher disturbance attenuation (lower state trajectory amplitude) of the proposed ETC designed for $(\tau, \gamma) = (1, 3.050)$ in comparison to the PTC designed for $(\tau, \gamma) = (4, \gamma^*_4 + \epsilon = 6.851)$ while it needs the same average transmission rate (for the given disturbance input w).

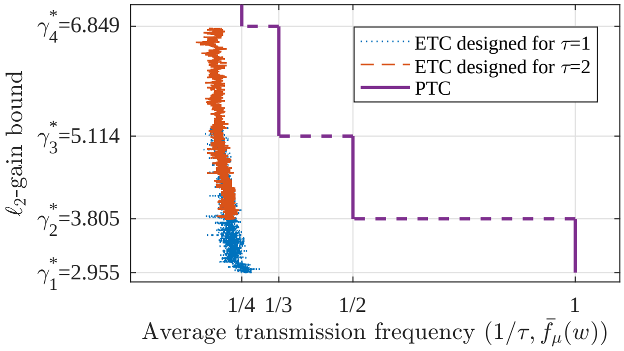

Fig. 3: Trade-off curves that result from the provided ℓ_2-consistent ETC designed for $\tau = 1$ and $\tau = 2$ and the given disturbance input w in comparison to the one that can be achieved by PTC.

Fig. 3 shows the trade-off curves between the ℓ_2-gain bound and the average transmission rate from the scheduler to the controller both for the proposed ℓ_2-consistent ETC and the designed periodic time-triggered ℓ_2-controller. The stair-type curve indicates the trade-off values one can achieve by following a periodic ℓ_2-controller. However, the dotted blue and the dashed red lines indicate the trade-off values resulting from the provided ℓ_2-consistent ETC for $\tau = 1$ and $\tau = 2$, respectively, for the given disturbance input and a time horizon of 250 and zero initial condition. As it can be seen, the proposed ℓ_2-consistent ETC can result in the same ℓ_2 performance level as the periodic ℓ_2-controller, however, with a much lower average transmission frequency. In other words, Fig. 3 shows that the proposed ℓ_2-consistent ETC can result in much better trade-off values between the ℓ_2-gain bound and the average transmission rate in comparison to PTC, thereby demonstrating the strengths of the proposed ETC algorithms.

V CONCLUSIONS

In this article, we first followed a game-theoretical approach to design a *periodic* time-triggered ℓ_2-controller for a given ℓ_2-gain bound γ and a discrete-time linear system with output feedback. The concept of *information state* [24] was employed throughout this controller design to define a minimax optimization problem, where the controller acts as a minimizer and the measured output is the maximizer of the cost function. Based on this setup, we proposed an event-triggered ℓ_2-controller, which guarantees the same ℓ_2-gain bound as the designed periodic time-triggered ℓ_2-controller, however, resulting in a lower or at most the same average data transmission rate from the sensors to the controller in comparison to that of the designed periodic time-triggered ℓ_2-controller. A numerical example showed that the proposed policy can, in fact, save considerable amount of communications in comparison to its periodic time-triggered ℓ_2-controller counterpart, while guaranteeing the same level of ℓ_2-gain bound for the control loop.

APPENDIX

A Proof of Lemma 1

The proof of this lemma is provided as complementary material due to the space limitation.

B Proof of Lemma 2

First we solve the finite-horizon zero-sum quadratic dynamic game (15), then we discuss the conditions under which the stationary version of the determined control policy is the solution of the 1-periodic ℓ_2-control problem according to Definition 2. First of all, we can easily show that $q(x_T) = x_T^{\intercal} X x_T$, where X follows (19) [22]. Now to solve (15), we follow a dynamic programming approach. Therefore, we should determine a value function at every $t \in \mathbb{N}_0^{T-1}$, i.e., $W_t : \mathbb{R}^n \to \mathbb{R}$, which satisfies

$$W_t(\bar{x}_t) := \max_{\{y_k := \mathcal{T}_k(\mathcal{E}_k)\}_{k\in\mathbb{N}_t^{T-1}}} \min_{\{u_k = \mathcal{R}_k^{\pi_\tau}(\mathcal{F}_k^{\pi_\tau})\}_{k\in\mathbb{N}_t^{T-1}}} \max_{x_T\in\mathbb{R}^n} [p_T(x_T) + x_T^{\intercal} X x_T], \tag{A.1}$$

for $\tau = 1$, where $\bar{x}_t$ follows (10) and

$$\begin{aligned}&W_T(\bar{x}_T) := \max\nolimits_{x_T\in\mathbb{R}^n}[p_T(x_T) + x_T^{\intercal} X x_T] = \max\nolimits_{x_T\in\mathbb{R}^n}\\ &[-\gamma^2(x_T - \bar{x}_T)^{\intercal} Y_T^{-1}(x_T - \bar{x}_T) + Z_T + x_T^{\intercal} X x_T]\\ &= \max\nolimits_{x_T\in\mathbb{R}^n}[-\gamma^2 x_T^{\intercal}(Y_T^{-1} - \gamma^{-2}X)x_T + 2\gamma^2 x_T^{\intercal} Y_T^{-1}\bar{x}_T]\\ &-\gamma^2\bar{x}_T^{\intercal} Y_T^{-1}\bar{x}_T + Z_T = \gamma^2\bar{x}_T^{\intercal} Y_T^{-1}(Y_T^{-1} - \gamma^{-2}X)^{-1}Y_T^{-1}\bar{x}_T\\ &-\gamma^2\bar{x}_T^{\intercal} Y_T^{-1}\bar{x}_T + Z_T = \bar{x}_T^{\intercal} X(I - \gamma^{-2}Y_T X)^{-1}\bar{x}_T + Z_T,\end{aligned} \tag{A.2}$$

under the condition that

$$\Omega_T := (I - \gamma^{-2}Y_T X)^{-1} \succ 0. \tag{A.3}$$

Note that if Ω_T has a negative eigenvalue, then $W_T(\bar{x}_T)$ is unbounded (necessity of the condition). We can also show that for any linear system, when the controller has access to the output vector at every $t \in \mathbb{N}_0^{T-1}$, then

$$W_t(\bar{x}_t) = \bar{x}_t^{\intercal} X_t \Omega_t \bar{x}_t + Z_t, \tag{A.4}$$

at all $t \in \mathbb{N}_0^T$, where $\Omega_t := (I - \gamma^{-2}Y_t X_t)^{-1}$, X_t is a positive definite matrix, which will be determined based on an iterative Riccati equation in the sequel, and Y_t, Z_t follow (9) and (11), respectively. Now for an arbitrary $i \in \mathbb{N}_0^{T-1}$,

$$\begin{aligned}\Delta W_i :&= W_{i+1}(\bar{x}_{i+1}) - W_i(\bar{x}_i) = \bar{x}_{i+1}^{\intercal} X_{i+1}\Omega_{i+1}\bar{x}_{i+1}\\ &+ Z_{i+1} - \bar{x}_i^{\intercal} X_i\Omega_i\bar{x}_i - Z_i = \bar{x}_i^{\intercal}\bar{Q}_i\bar{x}_i + \bar{u}_i^{\intercal}\bar{R}_i\bar{u}_i\\ &-\gamma^2\bar{y}_i^{\intercal}\bar{V}_i\bar{y}_i + \bar{x}_{i+1}^{\intercal} X_{i+1}\Omega_{i+1}\bar{x}_{i+1} - \bar{x}_i^{\intercal} X_i\Omega_i\bar{x}_i.\end{aligned}$$

We now substitute $\bar{x}_{i+1}$ based on (10) in the above equality

$$\begin{aligned}&\Delta W_i = \bar{x}_i^{\intercal}(\bar{Q}_i - X_i\Omega_i)\bar{x}_i + \bar{u}_i^{\intercal}\bar{R}_i\bar{u}_i - \gamma^2\bar{y}_i^{\intercal}\bar{V}_i\bar{y}_i\\ &+(\bar{A}_i\bar{x}_i + \bar{B}_i\bar{u}_i + \bar{D}_i\bar{y}_i)^{\intercal} X_{i+1}\Omega_{i+1}(\bar{A}_i\bar{x}_i + \bar{B}_i\bar{u}_i + \bar{D}_i\bar{y}_i)\\ &= \bar{x}_i^{\intercal}(\bar{Q}_i - X_i\Omega_i)\bar{x}_i - \gamma^2\bar{y}_i^{\intercal}\bar{V}_i\bar{y}_i + (\bar{u}_i - \bar{u}_i^*)^{\intercal}\\ &(\bar{R}_i + \bar{B}_i^{\intercal} X_{i+1}\Omega_{i+1}\bar{B}_i)(\bar{u}_i - \bar{u}_i^*)\\ &+(\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i)^{\intercal} X_{i+1}\Omega_{i+1}L_i(\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i),\end{aligned}$$

where

$$\begin{aligned}L_i :&= I - \bar{B}_i(\bar{R}_i + \bar{B}_i^{\intercal} X_{i+1}\Omega_{i+1}\bar{B}_i)^{-1}\bar{B}_i^{\intercal} X_{i+1}\Omega_{i+1}\\ &= I - \bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\intercal}\left(\Omega_{i+1}^{-1}X_{i+1}^{-1} + \bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\intercal}\right)^{-1}\\ &= (X_{i+1}\Omega_{i+1})^{-1}(X_{i+1}^{-1} - \gamma^{-2}Y_{i+1} + \bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\intercal})^{-1},\end{aligned}$$

in which the second equality is determined based on the following matrix inversion equality,

$$\mathcal{C}\mathcal{D}(\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{D})^{-1} = (\mathcal{C}^{-1} + \mathcal{D}\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{D}\mathcal{A}^{-1}.$$

Moreover, by considering

$$\begin{aligned}G_{i+1} &:= (X_{i+1}^{-1} - \gamma^{-2}Y_{i+1} + \bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\intercal})^{-1},\\ \Phi_i &:= \bar{R}_i + \bar{B}_i^{\intercal} X_{i+1}\Omega_{i+1}\bar{B}_i,\end{aligned} \tag{A.5}$$

we have

$$\begin{aligned}\bar{u}_i^* &= -\Phi_i^{-1}\bar{B}_i^{\intercal} X_{i+1}\Omega_{i+1}(\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i)\\ &= -\bar{R}_i^{-1}\bar{B}_i^{\intercal} G_{i+1}(\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i).\end{aligned} \tag{A.6}$$

Then, under the condition that $\Psi_i := \gamma^2\bar{V}_i - \bar{D}_i^{\intercal} G_{i+1}\bar{D}_i \succ 0$,

$$\begin{aligned}\Delta W_i &= \bar{x}_i^{\intercal}(\bar{Q}_i - X_i\Omega_i)\bar{x}_i - \gamma^2\bar{y}_i^{\intercal}\bar{V}_i\bar{y}_i + (\bar{u}_i - \bar{u}_i^*)^{\intercal}\Phi_i\\ &(\bar{u}_i - \bar{u}_i^*) + (\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i)^{\intercal} G_{i+1}(\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i)\\ &= \bar{x}_i^{\intercal}(\bar{Q}_i - X_i\Omega_i + \bar{A}_i^{\intercal} G_{i+1}F_i\bar{A}_i)\bar{x}_i + (\bar{u}_i - \bar{u}_i^*)^{\intercal}\Phi_i\\ &(\bar{u}_i - \bar{u}_i^*) - (\bar{y}_i - \bar{y}_i^*)^{\intercal}\Psi_i(\bar{y}_i - \bar{y}_i^*),\end{aligned} \tag{A.7}$$

where

$$\begin{aligned} F_i &:= I + \bar{D}_i(\gamma^2 \bar{V}_i - \bar{D}_i^\intercal G_{i+1} \bar{D}_i)^{-1} \bar{D}_i^\intercal G_{i+1} \\ &= G_{i+1}^{-1}(G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal)^{-1}, \\ \bar{y}_i^* &:= (\gamma^2 \bar{V}_i - \bar{D}_i^\intercal G_{i+1} \bar{D}_i)^{-1} \bar{D}_i^\intercal G_{i+1} \bar{A}_i \bar{x}_i. \end{aligned}$$

We can show that $\bar{V}_i \succ 0$ for all $i \in \mathbb{N}_0^{T-1}$ if $\Omega_i \succ 0$. For this purpose, we have

$$\begin{aligned} &C_2 \tilde{S}_i^{-1} C_2^\intercal N^{-1} = C_2(Y_i^{-1} + C_2^\intercal N^{-1} C_2 - \gamma^{-2}\tilde{Q})^{-1} C_2^\intercal N^{-1} \\ &= C_2(Y_i^{-1} - \gamma^{-2}\tilde{Q})^{-1} C_2^\intercal (N + C_2(Y_i^{-1} - \gamma^{-2}\tilde{Q})^{-1} C_2^\intercal)^{-1}, \end{aligned}$$

based on assumption that $N \succ 0$. Then by substituting the above equality into $\bar{V}_i$ defined in (12),

$$\bar{V}_i = \big(N + C_2(I - \gamma^{-2} Y_i \tilde{Q})^{-1} Y_i C_2^\intercal\big)^{-1},$$

which indicates that $\bar{V}_i \succ 0$ for all $i \in \mathbb{N}_0^{T-1}$, due to the fact that $I - \gamma^{-2} Y_i \tilde{Q} \succ I - \gamma^{-2} Y_i X_i$, and based on the assumption that $I - \gamma^{-2} Y_i X_i = \Omega_i^{-1} \succ 0$.
Now if $G_{i+1} \succ 0$, and $G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal \succ 0$, by using the matrix inversion lemma for $\Psi_i^{-1} D_i^\intercal G_{i+1}$ and Ψ_i^{-1}, we have

$$\bar{y}_i^* = \gamma^{-2} \bar{V}_i^{-1} \bar{D}_i^\intercal (G_{i+1}^{-1} - \gamma^{-2} \bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal)^{-1} \bar{A}_i \bar{x}_i, \tag{A.8}$$

and

$$\Psi_i^{-1} = \gamma^{-2} \bar{V}_i^{-1}(I + \gamma^{-2} \bar{D}_i^\intercal (G_{i+1}^{-1} - \gamma^{-2} \bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal)^{-1} \bar{D}_i \bar{V}_i^{-1}).$$

Therefore, based on the above equality, if $G_{i+1} \succ 0$, $G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal \succ 0$, and $\Omega_i \succ 0$, then $\Psi_i \succ 0$. We shall shortly see that $\Omega_i \succ 0$ is enough to guarantee $\Psi_i \succ 0$ (see the discussion after (A.11)). Now let us simplify $\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal = \alpha_i + 2\beta_i + \gamma_i$, where

$$\begin{aligned} \alpha_i &:= \tilde{A}\tilde{S}_i^{-1} C_2^\intercal N^{-1} \bar{V}_i^{-1} N^{-1} C_2 \tilde{S}_i^{-1} \tilde{A}^\intercal \\ &= \tilde{A}\tilde{S}_i^{-1} C_2^\intercal N^{-1}(I - C_2 \tilde{S}_i^{-1} C_2^\intercal N^{-1})^{-1} C_2 \tilde{S}_i^{-1} \tilde{A}^\intercal \\ &= \tilde{A}(\tilde{S}_i - C_2^\intercal N^{-1} C_2)^{-1} C_2^\intercal N^{-1} C_2 \tilde{S}_i^{-1} \tilde{A}^\intercal \\ &= \tilde{A}(Y_i^{-1} - \gamma^{-2}\tilde{Q})^{-1}\big(I - (Y_i^{-1} - \gamma^{-2}\tilde{Q})\tilde{S}_i^{-1}\big)\tilde{A}^\intercal \\ &= \tilde{A}\tilde{\Pi}_i^{-1}(Y_i - \tilde{\Pi}_i \tilde{S}_i^{-1})\tilde{A}^\intercal = \tilde{A}(\tilde{\Pi}_i^{-1} Y_i - \tilde{S}_i^{-1})\tilde{A}^\intercal, \\ \gamma_i &:= L\bar{V}_i L^\intercal = L N^{-1} L^\intercal - L N^{-1} C_2 \tilde{S}_i^{-1} C_2^\intercal N^{-1} L^\intercal, \\ \beta_i &:= \tilde{A}\tilde{S}_i^{-1} C_2^\intercal N^{-1} L^\intercal. \end{aligned}$$

Moreover, $Y_{i+1} = \hat{A} S_i^{-1} \hat{A}^\intercal + B_1 B_1^\intercal - L N^{-1} L^\intercal$. Therefore,

$$\begin{aligned} Y_{i+1} + \bar{D}_i \bar{V}_i^{-1} \bar{D}_i &= \hat{A} S_i^{-1} \hat{A}^\intercal + B_1 B_1^\intercal + 2\tilde{A}\tilde{S}_i^{-1} C_2^\intercal N^{-1} L^\intercal \\ &+ \tilde{A}(\tilde{\Pi}_i^{-1} Y_i - \tilde{S}_i^{-1})\tilde{A}^\intercal - L N^{-1} C_2 \tilde{S}_i^{-1} C_2^\intercal N^{-1} L^\intercal. \end{aligned} \tag{A.9}$$

Now, let us simplify $\bar{B}_i \bar{R}_i^{-1} \bar{B}_i^\intercal = \eta_i + 2\zeta_i + \xi_i$. We have

$$\bar{R}_i^{-1} = R^{-1} - \gamma^{-2} R^{-1} P^\intercal \tilde{S}_i^{-1} P R^{-1}.$$

Furthermore,

$$\begin{aligned} &\eta_i := B_2 \bar{R}_i^{-1} B_2^\intercal = B_2 R^{-1} B_2^\intercal - \gamma^{-2} B_2 R^{-1} P^\intercal \tilde{S}_i^{-1} P R^{-1} B_2^\intercal, \\ &\zeta_i := \gamma^{-2} B_2\big(R + \gamma^{-2} P^\intercal S_i^{-1} P\big)^{-1} P^\intercal S_i^{-1} \hat{A}^\intercal \\ &= \gamma^{-2} B_2 R^{-1} P^\intercal \tilde{S}_i^{-1} \hat{A}^\intercal, \ \xi_i := \gamma^{-4} \hat{A} S_i^{-1} P R^{-1} P^\intercal \tilde{S}_i^{-1} \hat{A}^\intercal. \end{aligned}$$

Finally, considering $H_i = I + (B_2 R^{-1} B_2^\intercal - \gamma^{-2} B_1 B_1^\intercal) X_{i+1}$,

$$\begin{aligned} &G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal = X_{i+1}^{-1} - \gamma^{-2} Y_{i+1} + \bar{B}_i \bar{R}_i^{-1} \bar{B}_i^\intercal \\ &- \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal = X_{i+1}^{-1} - \gamma^{-2} B_1 B_1^\intercal + B_2 R^{-1} B_2^\intercal \\ &- \gamma^{-2}\tilde{A}\tilde{\Pi}_i^{-1} Y_i \tilde{A}^\intercal = H_i X_{i+1}^{-1} - \gamma^{-2}\tilde{A}\tilde{\Pi}_i^{-1} Y_i \tilde{A}^\intercal. \end{aligned}$$

Therefore, by considering

$$X_i = \tilde{A}^\intercal X_{i+1} H_i^{-1} \tilde{A} + \tilde{Q}, \tag{A.10}$$

we have

$$\begin{aligned} (G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal)^{-1} &= (H_i X_{i+1}^{-1} - \gamma^{-2}\tilde{A}\tilde{\Pi}_i^{-1} Y_i \tilde{A}^\intercal)^{-1} \\ &= X_{i+1} H_i^{-1} + \gamma^{-2} X_{i+1} H_i^{-1} \tilde{A}(Y_i^{-1} - \gamma^{-2}\tilde{Q} \\ &- \gamma^{-2}(X_i - \tilde{Q}))^{-1} \tilde{A}^\intercal X_{i+1} H_i^{-1} \\ &= X_{i+1} H_i^{-1}(I + \gamma^{-2}\tilde{A}\Omega_i Y_i \tilde{A}^\intercal X_{i+1} H_i^{-1}). \end{aligned} \tag{A.11}$$

This indicates that if

$$\Omega_i \succ 0, \tag{A.12}$$

then $G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal \succ 0$. Based on (A.5), (A.12) is also enough to guarantee $G_{i+1} \succ 0$. Therefore, (A.12) is enough to guarantee $\Psi_i \succ 0$ for all $i \in \mathbb{N}_0$ (see the discussions after (A.8)). Moreover, based on (A.11),

$$\begin{aligned} &\bar{A}_i^\intercal G_{i+1} F_i \bar{A}_i = \bar{A}_i^\intercal (H_i X_{i+1}^{-1} - \gamma^{-2}\tilde{A}\tilde{\Pi}_i^{-1} Y_i \tilde{A}^\intercal)^{-1} \tilde{A}\tilde{\Pi}_i^{-1} \\ &= \tilde{\Pi}_i^{-\intercal} \tilde{A}^\intercal X_{i+1} H_i^{-1} \tilde{A}\Omega_i = X_i \Omega_i + \tilde{\Pi}_i^{-\intercal}\big(\tilde{A}^\intercal X_{i+1} H_i^{-1} \tilde{A} \\ &- (I - \gamma^{-2}\tilde{Q} Y_i) X_i\big)\Omega_i = -\tilde{Q}\tilde{\Pi}_i^{-1} + X_i \Omega_i = -\bar{Q}_i + X_i \Omega_i. \end{aligned} \tag{A.13}$$

Therefore, based on (A.7) and (A.13), we can conclude

$$\Delta W_i = (\bar{u}_i - \bar{u}_i^*)^\intercal \Phi_i (\bar{u}_i - \bar{u}_i^*) - (\bar{y}_i - \bar{y}_i^*)^\intercal \Psi_i (\bar{y}_i - \bar{y}_i^*),$$

and based on the definition of ΔW_i for all $i \in \mathbb{N}_0^{T-1}$, we have

$$W_T(\bar{x}_T) - W_0(\bar{x}_0) = \textstyle\sum_{i=0}^{T-1} \Delta W_i. \tag{A.14}$$

However, since $\bar{x}_0 = 0$ and $Z_0 = 0$, then $W_0(\bar{x}_0) = 0$. Moreover,

$$\begin{aligned} p_0(x_0) + \textstyle\sum_{i=0}^{T-1}[z_i^\intercal z_i - \gamma^2 w_i^\intercal w_i] &\leqslant p_T(x_T) + x_T^\intercal X_T x_T \\ &\leqslant W_T(\bar{x}_T), \end{aligned}$$

where the first inequality is based on Definition 3 and the second inequality is based on (A.2). Then by substituting $W_T(\bar{x}_T)$ from (A.14) into the above equality, we get

$$\textstyle\sum_{i=0}^{T-1}[z_i^\intercal z_i - \gamma^2 w_i^\intercal w_i] \leqslant -p_0(x_0) + \sum_{i=0}^{T-1} \Delta W_i. \tag{A.15}$$

Now let us simplify the control and disturbance policies. We have

$$(G_{i+1}^{-1} - \gamma^{-2}\bar{D}_i \bar{V}_i^{-1} \bar{D}_i^\intercal)^{-1} \bar{A}_i = X_{i+1} H_i^{-1} \tilde{A}\Omega_i. \tag{A.16}$$

Then based on (A.8), $\bar{y}_i^*$ is simplified as

$$\bar{y}_i^* = \gamma^{-2} \bar{V}_i^{-1} \bar{D}_i^\intercal X_{i+1} H_i^{-1} \tilde{A}\Omega_i \bar{x}_i.$$

We can also show that $\bar{V}_i^{-1} \bar{D}_i^\intercal = C_2 \tilde{\Pi}_i^{-1} Y_i \tilde{A}^\intercal + L^\intercal$. Then,

$$y_i^* := \bar{y}_i^* + \tilde{y}_i = (C_2 + \gamma^{-2} L^\intercal X_{i+1} H_i^{-1} \tilde{A})\Omega_i \bar{x}_i.$$

Moreover,

$$\bar{R}_i^{-1}\bar{B}_i^{\mathsf{T}} = R^{-1}B_2^{\mathsf{T}} + \gamma^{-2}R^{-1}P^{\mathsf{T}}\tilde{S}_i^{-1}\check{A}^{\mathsf{T}},$$
$$\bar{A}_i - \bar{D}_iC_2\tilde{\Pi}_i^{-1} = \check{A}\tilde{S}_i^{-1}Y_i^{-1},$$
$$\bar{A}_i\bar{x}_i + \bar{D}_i\bar{y}_i = \check{A}\tilde{S}_i^{-1}(Y_i^{-1}\bar{x}_i + C_2^{\mathsf{T}}N^{-1}y_i) + LN^{-1}y_i,$$

and

$$G_{i+1}^{-1} = X_{i+1}^{-1} - \gamma^{-2}Y_{i+1} + \bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\mathsf{T}} = X_{i+1}^{-1}$$
$$-\gamma^{-2}(B_1B_1^{\mathsf{T}} - LN^{-1}L^{\mathsf{T}}) + B_2R^{-1}B_2^{\mathsf{T}} - \gamma^{-2}\check{A}\tilde{S}_i^{-1}\check{A}^{\mathsf{T}}$$
$$= H_iX_{i+1}^{-1} + \gamma^{-2}LN^{-1}L^{\mathsf{T}} - \gamma^{-2}\check{A}\tilde{S}_i^{-1}\check{A}^{\mathsf{T}}.$$

Then by defining $\bar{x}_i^u = \tilde{S}_i^{-1}(Y_i^{-1}\bar{x}_i + C_2^{\mathsf{T}}N^{-1}y_i)$, we have

$$u_i^* = \check{u}_i + \tilde{u}_i + \bar{u}_i^* = -R^{-1}P^{\mathsf{T}}\bar{x}_i^u - R^{-1}(B_2^{\mathsf{T}} + \gamma^{-2}P^{\mathsf{T}}\tilde{S}_i^{-1}\check{A}^{\mathsf{T}})$$
$$(H_iX_{i+1}^{-1} + \gamma^{-2}LN^{-1}L^{\mathsf{T}} - \gamma^{-2}\check{A}\tilde{S}_i^{-1}\check{A}^{\mathsf{T}})^{-1}(\check{A}\bar{x}_i^u + LN^{-1}y_i). \tag{A.17}$$

Part i) For the infinite horizon problem, we have to consider the stationary version of the Riccati equations (A.10) and (9), where their solutions are X and Y, respectively, and take $Y_0 = Y$. Therefore, $Y_t = Y$ for all $t \in \mathbb{N}_0$ and instead of (A.3) and $S_t \succ 0$ for all $t \in \mathbb{N}_0$ and any $T \in \mathbb{N}$, $\gamma \in \mathbb{R}_{\succ 0}$ is needed to be selected such that

$$\gamma^2 I - YX \succ 0, \tag{A.18}$$

$$S = Y^{-1} + C_2^{\mathsf{T}}N^{-1}C_2 - \gamma^{-2}Q \succ 0. \tag{A.19}$$

In addition, it is necessary to have

$$\gamma^2 I - B_1^{\mathsf{T}}XB_1 \succ 0, \tag{A.20}$$

which guarantees $H = I + (B_2R^{-1}B_2^{\mathsf{T}} - \gamma^{-2}B_1B_1^{\mathsf{T}})X \succ 0$ following [26, Lemma 6.2]. One also needs to prove the necessity for the existence of X and Y (the stationary solutions for the Riccati equations (19) and (20)) for any $\gamma > \gamma_1^*$. If $X_t = \tilde{A}^{\mathsf{T}}X_{t+1}H_t^{-1}\tilde{A} + \tilde{Q}$ diverges as $t \to 0$, then the corresponding zero-sum minimax problem used for the determination of the state feedback ℓ_2-controller does not have a bounded upper value, see [22, Lemma 3.6], where in this case it is proved that γ is less than the ℓ_2-gain of the system [22]. Furthermore, if (9) diverges as $t \to \infty$, then $Y_t^{-1} \to 0$ and based on (A.1), $\max_{x_T \in \mathbb{R}^n}[p_T(x_T) + x_T^{\mathsf{T}}Xx_T] \to +\infty$ for a sufficiently large $T \in \mathbb{N}$. This will result in an unbounded $\hat{J}^*$ defined in (15). Now based on (A.15)

$$J \leqslant \gamma^2 x_0^{\mathsf{T}}Y^{-1}x_0 + \textstyle\sum_{i=0}^{\infty}\Delta\bar{W}_i, \tag{A.21}$$

where,

$$\Delta\bar{W}_i := (u_i - u_i^*)^{\mathsf{T}}\Phi(u_i - u_i^*) - (y_i - y_i^*)^{\mathsf{T}}\Psi(y_i - y_i^*)$$

in which Φ and Ψ are given in (30). Therefore, $u_i = u_i^*$ at all $i \in \mathbb{N}_0$ guarantees $J \leqslant \gamma^2 x_0^{\mathsf{T}}Y^{-1}x_0$. Moreover, given the observability of $(\tilde{Q}^{\frac{1}{2}}, \tilde{A})$ and the controllability of $(\hat{A}, B_1 - LN^{-1}D_{21})$, the solutions of the Riccati equations (19), (20), i.e. X and Y, under the conditions (A.18), (A.19) and (A.20), are positive definite. The proof of this argument is similar to the one presented in [22, Lemma 3.5].

Now we will show the *Lyapunov stability* of the control loop for $u_i = u_i^*$ at all $i \in \mathbb{N}_0$. For an arbitrary $t \in \mathbb{N}$, we have the following inequality based on Definition 3,

$$p_0(x_0) - p_t(x_t) \leqslant 0,$$

where $p_t(x_t) = -\gamma^2(x_t - \bar{x}_t)^{\mathsf{T}}Y^{-1}(x_t - \bar{x}_t) + Z_t$. Moreover, similar to (A.14), we can conclude

$$W_t(\bar{x}_t) - W_0(\bar{x}_0) = \textstyle\sum_{i=0}^{t-1}\Delta\bar{W}_i,$$

where $W_t(\bar{x}_t) = \bar{x}_t^{\mathsf{T}}X(I - \gamma^{-2}YX)^{-1}\bar{x}_t + Z_t$. Then by combining these two last expressions, we get

$$W_t(\bar{x}_t) - p_t(x_t) - W_0(\bar{x}_0) + p_0(x_0) \leqslant \textstyle\sum_{i=0}^{t-1}\Delta\bar{W}_i. \tag{A.22}$$

Now let us take $V(x_t, \bar{x}_t) := -p_t(x_t) + W_t(\bar{x}_t)$ as the Lyapunov function candidate for every $t \in \mathbb{N}_0$. Then based on (A.22), we have

$$V(x_t, \bar{x}_t) - V(x_0, \bar{x}_0) \leqslant \textstyle\sum_{i=0}^{t-1}\Delta\bar{W}_i. \tag{A.23}$$

Now if $u_i = u_i^*$ at all $i \in \mathbb{N}_0$, then $V(x_t, \bar{x}_t) - V(x_0, \bar{x}_0) \leqslant 0$ for all $t \in \mathbb{N}_0$, which proves that the control loop is Lyapunov stable.

Finally, we should prove the convergence of x_t to zero as time goes to infinity, when $w = (0, 0, \dots)$ and $u_i = u_i^*$ at all $i \in \mathbb{N}_0$. Based on (A.21), in this case

$$J = \textstyle\sum_{t=0}^{\infty} z_t^{\mathsf{T}}z_t \leqslant \gamma^2 x_0^{\mathsf{T}}Y^{-1}x_0.$$

Let us define $\nu_t^* := u_t^* + R^{-1}P^{\mathsf{T}}x_t$. Then

$$z_t^{\mathsf{T}}z_t = x_t^{\mathsf{T}}Qx_t + 2x^{\mathsf{T}}Pu_t^* + u_t^{*\mathsf{T}}Ru_t^* = x_t^{\mathsf{T}}\tilde{Q}x_t + \nu_t^{*\mathsf{T}}R\nu_t^*,$$

and $x_{t+1} = Ax_t + B_2u_t^* = \tilde{A}x_t + B_2\nu_t^*$. Now due to the boundedness of J, we can conclude that $\tilde{Q}^{\frac{1}{2}}x_t \to 0$ and $\nu_t^* \to 0$ as time converges to infinity. Then based on the observability of $(\tilde{Q}^{\frac{1}{2}}, \tilde{A})$ we can show that $x_t \to 0$ as $t \to \infty$.

Part ii) This part is already proved above, see (A.21).

C Proof of Lemma 3

Let us first introduce another formula for the dynamics of $\bar{x}_t$ and Z_t, besides (10) and (11). Consider $\bar{x}_0 = 0$ and $Z_0 = 0$, then for all $t \in \mathbb{N}_0$,

$$\bar{x}_{t+1} = \bar{A}_t\bar{x}_t + \hat{B}_t\hat{u}_t + \hat{D}_t\hat{y}_t, \tag{A.24}$$

$$Z_{t+1} = Z_t + \bar{x}_t^{\mathsf{T}}\bar{Q}_t\bar{x}_t + \hat{u}_t^{\mathsf{T}}\hat{R}_t\hat{u}_t - \gamma^2\hat{y}_t^{\mathsf{T}}\hat{V}_t\hat{y}_t, \tag{A.25}$$

where $\bar{A}_t, \bar{Q}_t$ and $\tilde{\Pi}_t$ are as given in (12), $\Pi_t := I - \gamma^{-2}Y_tQ$ and

$$\hat{B}_t = B_2 + \gamma^{-2}A\Pi_t^{-1}Y_tP, \quad \hat{D}_t = AS_t^{-1}C_2^{\mathsf{T}}N^{-1} + L\hat{V}_t,$$
$$\hat{R}_t = R + \gamma^{-2}P^{\mathsf{T}}\Pi_t^{-1}Y_tP, \quad \hat{V}_t = N^{-1}(I - C_2S_t^{-1}C_2^{\mathsf{T}}N^{-1}). \tag{A.26}$$

Moreover, at all $t \in \mathbb{N}_0$, $\hat{u}_t = u_t - \tilde{\tilde{u}}_t$ and $\hat{y}_t = y_t - (\tilde{\tilde{y}}_t + \check{\check{y}}_t)$ for

$$\tilde{\tilde{u}}_t = -\tilde{\tilde{K}}_t\bar{x}_t, \qquad \tilde{\tilde{y}}_t = \tilde{\tilde{L}}_t\bar{x}_t, \qquad \check{\check{y}}_t = \check{\check{L}}_tu_t, \tag{A.27}$$

where $\tilde{\tilde{K}}_t = R^{-1}P^{\mathsf{T}}\tilde{\Pi}_t^{-1}$, $\tilde{\tilde{L}}_t = C_2\Pi_t^{-1}$, $\check{\check{L}}_t = \gamma^{-2}C_2\Pi_t^{-1}Y_tP$. We can prove the above following the same steps as the one for (10) and (11).

Now when the information sets of the output generator and the controller follow (16) and (4), respectively, for a given $\tau \in \mathbb{N}$, then the value function defined in (A.1) at all transmission time-steps to the controller, i.e., when $\ell = (\kappa+1)\tau \in \mathbb{N}_0^{T-1}$, follows

$$W_\ell(\bar{x}_\ell) = \bar{x}_\ell^\intercal \bar{X}_\ell \bar{x}_\ell + Z_\ell, \tag{A.28}$$

for $\bar{X}_\ell = X_\ell \Omega_\ell$ and $\Omega_\ell := (I - \gamma^{-2} Y_\ell X_\ell)^{-1}$, where X_ℓ is an appropriate positive definite matrix determined based on (A.10) and Y_ℓ, Z_ℓ follow (9) and (11) (or (A.25)), respectively. Now we will determine the value functions at non-transmission time steps. Consider $\iota = \kappa\tau \in \mathbb{N}_0$ as the last transmission time-step before $\ell = (\kappa+1)\tau$. Then, based on Isaacs equation, we have

$$W_\iota(\bar{x}_\iota) = \max_{\bar{y}_\iota \in \mathbb{R}^o} \min_{U_\kappa \in \mathbb{R}^{m\tau}} \max_{\hat{y}_{\iota+1} \in \mathbb{R}^o} \cdots \max_{\hat{y}_{\ell-1} \in \mathbb{R}^o} W_\ell(\bar{x}_\ell).$$

Let us denote $r = \ell - 1$, then

$$\begin{aligned}
\hat{W}_r(\bar{x}_r, \hat{u}_r) &:= \max_{\hat{y}_r \in \mathbb{R}^o} \bar{x}_\ell^\intercal \bar{X}_\ell \bar{x}_\ell + Z_\ell \\
&= \max_{\hat{y}_r \in \mathbb{R}^o} (\bar{A}_r \bar{x}_r + \hat{B}_r \hat{u}_r + \hat{D}_r \hat{y}_r)^\intercal \bar{X}_\ell (\bar{A}_r \bar{x}_r + \hat{B}_r \hat{u}_r + \hat{D}_r \hat{y}_r) \\
&\quad + Z_r + \bar{x}_r^\intercal \bar{Q}_r \bar{x}_r + \hat{u}_r^\intercal \hat{R}_r \hat{u}_r - \gamma^2 \hat{y}_r^\intercal \hat{V}_r \hat{y}_r,
\end{aligned}$$

which results in

$$\hat{y}_r^* := \left(\gamma^2 \hat{V}_r - \hat{D}_r^\intercal \bar{X}_\ell \hat{D}_r\right)^{-1} \hat{D}_r^\intercal \bar{X}_\ell (\bar{A}_r \bar{x}_r + \hat{B}_r \hat{u}_r),$$

under the condition that

$$\gamma^2 \hat{V}_r - \hat{D}_r^\intercal \bar{X}_\ell \hat{D}_r \succ 0. \tag{A.29}$$

Note that if $\gamma^2 \hat{V}_r - \hat{D}_r^\intercal \bar{X}_\ell \hat{D}_r$ has a negative eigenvalue, then $\hat{W}_r(\bar{x}_r, \hat{u}_r)$ is unbounded. Now let us define $\Xi_\ell := (\bar{X}_\ell^{-1} - \gamma^{-2} \hat{D}_r \hat{V}_r^{-1} \hat{D}_r^\intercal)^{-1}$, then by substituting $\hat{y}_r^*$ into the cost function, it results in

$$\begin{aligned}
\hat{W}_r(\bar{x}_r, u_r) &= Z_r + \bar{x}_r^\intercal \bar{Q}_r \bar{x}_r + (u_r - \tilde{\tilde{u}}_r)^\intercal \hat{R}_r (u_r - \tilde{\tilde{u}}_r) \\
&\quad + (\bar{A}_r \bar{x}_r - \hat{B}_r \tilde{\tilde{u}}_r + \hat{B}_r u_r)^\intercal \Xi_\ell (\bar{A}_r \bar{x}_r - \hat{B}_r \tilde{\tilde{u}}_r + \hat{B}_r u_r).
\end{aligned}$$

For an arbitrary $r \in \mathbb{N}_0^{T-1}$, we have the following equalities,

$$\begin{aligned}
&\bar{A}_r \bar{x}_r - \hat{B}_r \tilde{\tilde{u}}_r = A \Pi_r^{-1} \bar{x}_r = \bar{\bar{A}}_r \bar{x}_r, \\
&\tilde{\tilde{u}}_r^\intercal \hat{R}_r \tilde{\tilde{u}}_r = \bar{x}_r \tilde{\Pi}_r^{-\intercal} P R^{-1} P^\intercal \Pi_r^{-1} \bar{x}_r^\intercal, \\
&\tilde{\Pi}_r^{-\intercal} P R^{-1} P^\intercal \Pi_r^{-1} + \bar{Q}_r = Q \Pi_r^{-1} = \bar{\bar{Q}}_r, \\
&u_r \hat{R}_r \tilde{\tilde{u}}_r = -u_r P^\intercal \Pi_r^{-1} \bar{x}_r = -u_r \bar{\bar{P}}_r^\intercal \bar{x}_r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{W}_r(\bar{x}_r, u_r) &= (\bar{\bar{A}}_r \bar{x}_r + \hat{B}_r u_r)^\intercal \Xi_\ell (\bar{\bar{A}}_r \bar{x}_r + \hat{B}_r u_r) \\
&\quad + Z_r + \bar{x}_r^\intercal \bar{\bar{Q}}_r \bar{x}_r + u_r^\intercal \hat{R}_r u_r + 2 u_r \bar{\bar{P}}_r^\intercal \bar{x}_r \\
&= \bar{x}_r^\intercal \bar{\Theta}_r \bar{x}_r + 2 u_r^\intercal \bar{N}_r \bar{x}_r + u_r^\intercal \bar{\Upsilon}_r u_r + Z_r,
\end{aligned}$$

where

$$\begin{aligned}
&\bar{\Theta}_r := \bar{\bar{A}}_r^\intercal \Xi_\ell \bar{\bar{A}}_r + \bar{\bar{Q}}_r, \bar{N}_r := \hat{B}_r^\intercal \Xi_\ell \bar{\bar{A}}_r + \bar{\bar{P}}_r^\intercal, \\
&\bar{\Upsilon}_r := \hat{R}_r + \hat{B}_r^\intercal \Xi_\ell \hat{B}_r.
\end{aligned}$$

Moreover, for any $i \in \mathbb{N}_0^{T-1}$, $\hat{D}_i \hat{V}_i^{-1} \hat{D}_i^\intercal = \alpha_i + 2\beta_i + \gamma_i$, where

$$\begin{aligned}
\alpha_i :&= A S_i^{-1} C_2^\intercal N^{-1} \hat{V}_i^{-1} N^{-1} C_2 S_i^{-1} A^\intercal \\
&= A S_i^{-1} C_2^\intercal N^{-1} (I - C_2 S_i^{-1} C_2^\intercal N^{-1})^{-1} C_2 S_i^{-1} A^\intercal \\
&= A (S_i - C_2^\intercal N^{-1} C_2)^{-1} C_2^\intercal N^{-1} C_2 S_i^{-1} A^\intercal \\
&= A (Y_i^{-1} - \gamma^{-2} Q)^{-1} \big(I - (Y_i^{-1} - \gamma^{-2} Q) S_i^{-1}\big) A^\intercal \\
&= A \Pi_i^{-1} (Y_i - \Pi_i S_i^{-1}) A^\intercal = A (\Pi_i^{-1} Y_i - S_i^{-1}) A^\intercal, \\
\gamma_i :&= L \hat{V}_i L^\intercal = L N^{-1} L^\intercal - L N^{-1} C_2 S_i^{-1} C_2^\intercal N^{-1} L^\intercal, \\
\beta_i :&= A S_i^{-1} C_2^\intercal N^{-1} L^\intercal.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Y_{i+1} + \hat{D}_i \hat{V}_i^{-1} \hat{D}_i^\intercal &= \hat{A} S_i^{-1} \hat{A}^\intercal + B_1 B_1^\intercal - L N^{-1} L^\intercal \\
&\quad + A (\Pi_i^{-1} Y_i - S_i^{-1}) A^\intercal - L N^{-1} C_2 S_i^{-1} C_2^\intercal N^{-1} L^\intercal \\
&\quad + L N^{-1} L^\intercal + 2 A S_i^{-1} C_2^\intercal N^{-1} L^\intercal = A \Pi_i^{-1} Y_i A^\intercal + B_1 B_1^\intercal.
\end{aligned} \tag{A.30}$$

Moreover, based on assumption $N \succ 0$, and

$$\begin{aligned}
C_2 S_t^{-1} C_2^\intercal N^{-1} &= C_2 (Y_t^{-1} + C_2^\intercal N^{-1} C_2 - \gamma^{-2} Q)^{-1} C_2^\intercal N^{-1} \\
&= C_2 (Y_t^{-1} - \gamma^{-2} Q)^{-1} C_2^\intercal (N + C_2 (Y_t^{-1} - \gamma^{-2} Q)^{-1} C_2^\intercal)^{-1}.
\end{aligned}$$

Then by substituting the above equality into $\hat{V}_t$ defined in (A.26),

$$\hat{V}_t = \left(N + C_2 (Y_t^{-1} - \gamma^{-2} Q)^{-1} C_2^\intercal\right)^{-1},$$

which proves that $\hat{V}_t \succ 0$ for all $t \in \mathbb{N}_0^{T-1}$, due to the fact that $Y_t^{-1} - \gamma^{-2} Q$ is assumed to be positive definite. Consequently, we can show that

$$\begin{aligned}
\Xi_\ell^{-1} &= \bar{X}_\ell^{-1} - \gamma^{-2} \hat{D}_r \hat{V}_r^{-1} \hat{D}_r^\intercal = X_\ell^{-1} - \gamma^{-2} Y_\ell \\
&\quad - \gamma^{-2} \hat{D}_r \hat{V}_r^{-1} \hat{D}_r^\intercal = X_\ell^{-1} - \gamma^{-2} (B_1 B_1^\intercal + A \Pi_r^{-1} Y_r A^\intercal)
\end{aligned}$$

in which the second equality is based on (A.30). Then we can conclude

$$\begin{aligned}
\bar{\Theta}_r &= \Pi_r^{-\intercal} A^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal - \gamma^{-2} A \Pi_r^{-1} Y_r A^\intercal)^{-1} A \Pi_r^{-1} \\
&\quad + Q \Pi_r^{-1} = \Pi_r^{-\intercal} A^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A \big(I - \gamma^{-2} Y_r Q \\
&\quad - \gamma^{-2} Y_r A^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A\big)^{-1} + \Pi_r^{-\intercal} Q,
\end{aligned}$$

where by considering

$$\Theta_r = A^\intercal X_\ell (I - \gamma^{-2} B_1 B_1^\intercal X_\ell)^{-1} A + Q, \tag{A.31}$$

and $\Omega_r = (I - \gamma^{-2} Y_r \Theta_r)^{-1}$,

$$\begin{aligned}
\bar{\Theta}_r &= \Pi_r^{-\intercal} (\Theta_r - Q) \big(I - \gamma^{-2} Y_r Q - \gamma^{-2} Y_r (\Theta_r - Q)\big)^{-1} \\
&\quad + \Pi_r^{-\intercal} Q = \Pi_r^{-\intercal} (\Theta_r - Q) \Omega_r + \Pi_r^{-\intercal} Q = \Theta_r \Omega_r.
\end{aligned}$$

Now we will simplify $\bar{N}_r$. Based on previous equation, we have $\Xi_\ell \bar{\bar{A}}_r = (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A \Omega_r$, and

$$\hat{B}_r^\intercal \Xi_\ell \bar{\bar{A}}_r = \big(B_2 + \gamma^{-2} A \Pi_r^{-1} Y_r P\big)^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A \Omega_r.$$

Moreover,

$$\begin{aligned}
&\gamma^{-2} P^\intercal \Pi_r^{-1} Y_r A^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A \Omega_r + P^\intercal \Pi_r^{-1} = P^\intercal \\
&\Pi_r^{-1} \big(\gamma^{-2} Y_r A^\intercal (X_\ell^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A + I - \gamma^{-2} Y_r \Theta_r\big) \Omega_r \\
&= P^\intercal \Pi_r^{-1} \big(\gamma^{-2} Y_r (\Theta_r - Q) + I - \gamma^{-2} Y_r \Theta_r\big) \Omega_r = P^\intercal \Omega_r.
\end{aligned}$$

Therefore,

$$\bar{N}_r = \big(B_2^\intercal(X_\ell^{-1} - \gamma^{-2}B_1B_1^\intercal)^{-1}A + P^\intercal\big)\Omega_r. \tag{A.32}$$

Now by an induction argument, we assume that at an arbitrary time step t, where $\iota \prec t \prec \ell$, the value function follows

$$\hat{W}_t(\bar{x}_t, U_t) = \bar{x}_t^\intercal\bar{\Theta}_t\bar{x}_t + 2U_t^\intercal\bar{N}_t\bar{x}_t + U_t^\intercal\bar{\Upsilon}_tU_t + Z_t, \tag{A.33}$$

where $U_t = [u_t^\intercal \dots u_r^\intercal]^\intercal$. Consider $s = t-1$, then

$$\begin{aligned}
&\hat{W}_s(\bar{x}_s, U_s) := \max_{\hat{y}_s\in\mathbb{R}^o}\big[\bar{x}_t^\intercal\bar{\Theta}_t\bar{x}_t + 2U_t^\intercal\bar{N}_t\bar{x}_t + U_t^\intercal\bar{\Upsilon}_tU_t + Z_t\big]\\
&= \max_{\hat{y}_s\in\mathbb{R}^o}\big[(\bar{A}_s\bar{x}_s + \hat{B}_s\hat{u}_s + \hat{D}_s\hat{y}_s)^\intercal\bar{\Theta}_t(\bar{A}_s\bar{x}_s + \hat{B}_s\hat{u}_s + \hat{D}_s\hat{y}_s)\\
&+ 2U_t^\intercal\bar{N}_t(\bar{A}_s\bar{x}_s + \hat{B}_s\hat{u}_s + \hat{D}_s\hat{y}_s) + U_t^\intercal\bar{\Upsilon}_tU_t\\
&+ Z_s + \bar{x}_s^\intercal\bar{Q}_s\bar{x}_s + \hat{u}_s^\intercal\hat{R}_s\hat{u}_s - \gamma^2\hat{y}_s^\intercal\hat{V}_s\hat{y}_s\big].
\end{aligned}$$

Thus, under the condition that

$$\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s \succ 0, \tag{A.34}$$

we have

$$\hat{y}_s^* = (\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s)^{-1}\hat{D}_s^\intercal\big(\bar{\Theta}_t(\bar{A}_s\bar{x}_s + \hat{B}_s\hat{u}_s) + \bar{N}_t^\intercal U_t\big). \tag{A.35}$$

It is worth mentioning that if $\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s$ has a negative eigenvalue, then $\hat{W}_s(\bar{x}_s, U_s)$ is unbounded (necessity of the condition (A.34)). Now by substituting $\hat{y}_s^*$ into $\hat{W}_s(\bar{x}_s, U_s)$ and having

$$\begin{aligned}
&\bar{\bar{A}}_s := A\Pi_s^{-1}, \quad \bar{\bar{Q}}_s := Q\Pi_s^{-1}, \quad \bar{\bar{P}}_s^\intercal := P^\intercal\Pi_s^{-1},\\
&\Xi_t := (\bar{\Theta}_t^{-1} - \gamma^{-2}\hat{D}_s\hat{V}_s^{-1}\hat{D}_s^\intercal)^{-1},
\end{aligned}$$

we get

$$\begin{aligned}
\hat{W}_s(\bar{x}_s,U_s) &= (\bar{\bar{A}}_s\bar{x}_s + \hat{B}_su_s)^\intercal\Xi_t(\bar{\bar{A}}_s\bar{x}_s + \hat{B}_su_s)\\
&+ 2(\bar{\bar{A}}_s\bar{x}_s + \hat{B}_su_s)^\intercal\Xi_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal U_t\\
&+ U_t^\intercal\big(\bar{\Upsilon}_t + \bar{N}_t\hat{D}_s\big(\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s\big)^{-1}\hat{D}_s^\intercal\bar{N}_t^\intercal\big)U_t\\
&+ Z_s + \bar{x}_s^\intercal\bar{\bar{Q}}_s\bar{x}_s + u_s^\intercal\hat{R}_su_s + 2u_s^\intercal\bar{\bar{P}}_s^\intercal\bar{x}_s\\
&= \bar{x}_s^\intercal\bar{\Theta}_s\bar{x}_s + 2U_s^\intercal\bar{N}_s\bar{x}_s + U_s^\intercal\bar{\Upsilon}_sU_s + Z_s,
\end{aligned}$$

where

$$\begin{aligned}
&\bar{\Theta}_s = \bar{\bar{Q}}_s + \bar{\bar{A}}_s^\intercal\Xi_t\bar{\bar{A}}_s,\\
&\bar{N}_s = \begin{bmatrix}\hat{B}_s^\intercal\\ \bar{N}_t\bar{\Theta}_t^{-1}\end{bmatrix}\Xi_t\bar{\bar{A}}_s + \begin{bmatrix}\bar{\bar{P}}_s^\intercal\\ 0\end{bmatrix},\ \bar{\Upsilon}_s = \begin{bmatrix}\mathcal{P}_{11} & \mathcal{P}_{21}^\intercal\\ \mathcal{P}_{21} & \mathcal{P}_{22}\end{bmatrix},
\end{aligned} \tag{A.36}$$

in which

$$\begin{aligned}
&\mathcal{P}_{11} = \hat{R}_s + \hat{B}_s^\intercal\Xi_t\hat{B}_s, \qquad \mathcal{P}_{21} = \bar{N}_t\bar{\Theta}_t^{-1}\Xi_t\hat{B}_s,\\
&\mathcal{P}_{22} = \bar{\Upsilon}_t + \gamma^{-2}\bar{N}_t\hat{D}_s\hat{V}_s^{-1}\hat{D}_s^\intercal\Xi_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal.
\end{aligned}$$

We can also simplify $\mathcal{P}_{22}$ as follows

$$\begin{aligned}
\mathcal{P}_{22} &= \bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal + \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal + \gamma^{-2}\bar{N}_t\hat{D}_s\hat{V}_s^{-1}\hat{D}_s^\intercal\\
&\Xi_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal = \bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal + \bar{N}_t\bar{\Theta}_t^{-1}\Xi_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal.
\end{aligned}$$

Therefore, $\hat{W}_s(\bar{x}_s, U_s)$ has the same form as $\hat{W}_t(\bar{x}_t, U_t)$ and the assumption of the induction (A.33) is correct. Then by considering $L_s = I - \gamma^{-2}B_1B_1^\intercal\Theta_t$, and

$$\Theta_s = A^\intercal\Theta_tL_s^{-1}A + Q, \tag{A.37}$$

we can show that $\bar{\Theta}_s = \Theta_s\Omega_s$. Moreover,

$$\Xi_t^{-1} = \Theta_t^{-1} - \gamma^{-2}(B_1B_1^\intercal + A\Pi_s^{-1}Y_sA^\intercal). \tag{A.38}$$

Now if we assume $\iota = t-1$, then

$$\begin{aligned}
&\hat{W}_\iota(\bar{x}_\iota, \bar{y}_\iota) = \min_{\hat{U}_\iota\in\mathbb{R}^{m\tau}}\big[\bar{x}_t^\intercal\bar{\Theta}_t\bar{x}_t + 2U_t^\intercal\bar{N}_t\bar{x}_t + U_t^\intercal\bar{\Upsilon}_tU_t + Z_t\big]\\
&= \min_{\hat{U}_\iota\in\mathbb{R}^{m\tau}}\big[(\bar{A}_\iota\bar{x}_\iota + \bar{B}_\iota\bar{u}_\iota + \bar{D}_\iota\bar{y}_\iota)^\intercal\bar{\Theta}_t(\bar{A}_\iota\bar{x}_\iota + \bar{B}_\iota\bar{u}_\iota + \bar{D}_\iota\bar{y}_\iota)\\
&+ 2U_t^\intercal\bar{N}_t(\bar{A}_\iota\bar{x}_\iota + \bar{B}_\iota\bar{u}_\iota + \bar{D}_\iota\bar{y}_\iota) + U_t^\intercal\bar{\Upsilon}_tU_t\\
&+ Z_\iota + \bar{x}_\iota^\intercal\bar{Q}_\iota\bar{x}_\iota + \bar{u}_\iota^\intercal\bar{R}_\iota\bar{u}_\iota - \gamma^2\bar{y}_\iota^\intercal\bar{V}_\iota\bar{y}_\iota\big],
\end{aligned}$$

where $\hat{U}_\iota = [\bar{u}_\iota^\intercal \quad U_t^\intercal]^\intercal$. This results in

$$\hat{U}_\iota^* = -\bar{K}_\iota(\bar{A}_\iota\bar{x}_\iota + \bar{D}_\iota\bar{y}_\iota), \tag{A.39}$$

where

$$\bar{K}_\iota := \begin{bmatrix}\bar{R}_\iota + \bar{B}_\iota^\intercal\bar{\Theta}_t\bar{B}_\iota & \bar{B}_\iota^\intercal\bar{N}_t^\intercal\\ \bar{N}_t\bar{B}_\iota & \bar{\Upsilon}_t\end{bmatrix}^{-1}\begin{bmatrix}\bar{B}_\iota^\intercal\bar{\Theta}_t\\ \bar{N}_t\end{bmatrix}.$$

Moreover, let us introduce the following matrices

$$\mathcal{A} = \begin{bmatrix}\bar{R}_\iota & 0\\ 0 & \bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal\end{bmatrix},\ \ \mathcal{B} = \begin{bmatrix}\bar{B}_\iota^\intercal\\ \bar{N}_t\bar{\Theta}_t^{-1}\end{bmatrix},\ \ \mathcal{C} = \bar{\Theta}_t,$$

then

$$\begin{aligned}
\bar{K}_\iota &= (\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal)^{-1}\mathcal{B}\mathcal{C} = \mathcal{A}^{-1}\mathcal{B}(\mathcal{C}^{-1} + \mathcal{B}^\intercal\mathcal{A}^{-1}\mathcal{B})^{-1}\\
&= \begin{bmatrix}\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal\\ (\bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal)^{-1}\bar{N}_t\bar{\Theta}_t^{-1}\end{bmatrix}\big(\bar{\Theta}_t^{-1} + \bar{B}_\iota\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal\\
&+ \bar{\Theta}_t^{-1}\bar{N}_t^\intercal(\bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal)^{-1}\bar{N}_t\bar{\Theta}_t^{-1}\big)^{-1}.
\end{aligned}$$

We can use the following two inequalities

$$\begin{aligned}
&(\bar{\Upsilon}_t - \bar{N}_t\bar{\Theta}_t^{-1}\bar{N}_t^\intercal)^{-1}\bar{N}_t\bar{\Theta}_t^{-1} = \bar{\Upsilon}_t^{-1}\bar{N}_t\bar{X}_t^{-1},\\
&\bar{\Theta}_t^{-1} + \bar{\Theta}_t^{-1}\bar{N}_t^\intercal\bar{\Upsilon}_t^{-1}\bar{N}_t\bar{X}_t^{-1} = \bar{X}_t^{-1},
\end{aligned} \tag{A.40}$$

where $\bar{X}_t = \bar{\Theta}_t - \bar{N}_t^\intercal\bar{\Upsilon}_t^{-1}\bar{N}_t$. Then

$$\bar{K}_\iota = \begin{bmatrix}\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal\\ \bar{\Upsilon}_t^{-1}\bar{N}_t\bar{X}_t^{-1}\end{bmatrix}G_t, \tag{A.41}$$

where $G_t = (\bar{X}_t^{-1} + \bar{B}_\iota\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal)^{-1}$. Moreover, by replacing $\hat{U}_\iota$ with $\hat{U}_\iota^*$ in $\hat{W}_\iota(\bar{x}_\iota, \bar{y}_\iota)$, we get

$$\begin{aligned}
W_\iota(\bar{x}_\iota) := \max_{\bar{y}_\iota\in\mathbb{R}^0}\big[&Z_\iota + \bar{x}_\iota^\intercal\bar{Q}_\iota\bar{x}_\iota - \gamma^2\bar{y}_\iota^\intercal\bar{V}_\iota\bar{y}_\iota\\
&+ (\bar{A}_\iota\bar{x}_\iota + \bar{D}_\iota\bar{y}_\iota)^\intercal G_t(\bar{A}_\iota\bar{x}_\iota + \bar{D}_\iota\bar{y}_\iota)\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\bar{y}_\iota^* = \big(\gamma^2\bar{V}_\iota - \bar{D}_\iota^\intercal G_t\bar{D}_\iota\big)^{-1}\bar{D}_\iota^\intercal G_t\bar{A}_\iota\bar{x}_\iota\\
&= \gamma^{-2}\bar{V}_\iota^{-1}\bar{D}_\iota^\intercal(\bar{X}_t^{-1} + \bar{B}_\iota\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal - \gamma^{-2}\bar{D}_\iota\bar{V}_\iota^{-1}\bar{D}_\iota^\intercal)^{-1}\bar{A}_\iota\bar{x}_\iota
\end{aligned}$$

if $\gamma^2\bar{V}_\iota - \bar{D}_\iota^\intercal G_t\bar{D}_\iota \succ 0$. As we showed in the proof of Lemma 2, this condition is equivalent to $\mathcal{S}_\iota := \bar{X}_t^{-1} - \gamma^{-2}\bar{D}_\iota\bar{V}_\iota^{-1}\bar{D}_\iota^\intercal + \bar{B}_\iota\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal \succ 0$. Moreover, similar to (A.11), we can show that $\mathcal{S}_\iota \succ 0$ if

$$\gamma^2I - Y_\iota X_\iota \succ 0. \tag{A.42}$$

Therefore,

$$\bar{y}_\iota^* = \gamma^{-2}\bar{V}_\iota^{-1}\bar{D}_\iota^\intercal\mathcal{S}_\iota^{-1}\bar{A}_\iota\bar{x}_\iota.$$

Moreover, assuming

$$X_\iota = \tilde{A}(X_{\iota+1}^{-1} - \gamma^{-2}B_1B_1^\intercal + B_2R^{-1}B_2^\intercal)^{-1}\tilde{A} + \tilde{Q}, \quad \text{(A.43)}$$

then similar to (A.16),

$$\mathcal{S}_\iota^{-1}\bar{A}_\iota = (X_{\iota+1}^{-1} - \gamma^{-2}B_1B_1^\intercal + B_2R^{-1}B_2^\intercal)^{-1}\tilde{A}(I - \gamma^{-2}Y_\iota X_\iota)^{-1}.$$

Therefore,

$$y_\iota^* = \gamma^{-2}(C_2\tilde{\Pi}_\iota^{-1}Y_\iota\tilde{A}^\intercal + L^\intercal)\tilde{A}(I - \gamma^{-2}Y_\iota X_\iota)^{-1}\bar{x}_\iota + C_2 \tilde{\Pi}_\iota^{-1}\bar{x}_\iota = \big(C_2 + \gamma^{-2}L^\intercal X_{\iota+1}H_\iota^{-1}\tilde{A}\big)(I - \gamma^{-2}Y_\iota X_\iota)^{-1}\bar{x}_\iota. \quad \text{(A.44)}$$

Moreover, based on (A.39) and (A.41),

$$\bar{u}_\iota^* = -\bar{R}_\iota^{-1}\bar{B}_\iota^\intercal G_t(\bar{A}_\iota\bar{x}_\iota + \bar{D}_\iota\bar{y}_\iota).$$

Then,

$$\begin{aligned} u_\iota^* = \check{u}_\iota + \tilde{u}_\iota + \bar{u}_\iota^* = -R^{-1}P^\intercal\bar{x}_\iota^u - R^{-1} \\ (B_2^\intercal + \gamma^{-2}P^\intercal\tilde{S}_\iota^{-1}\check{A}^\intercal)(H_\iota X_{\iota+1}^{-1} + \gamma^{-2}LN^{-1}L^\intercal \\ - \gamma^{-2}\check{A}\tilde{S}_\iota^{-1}\check{A}^\intercal)^{-1}(\check{A}\bar{x}_\iota^u + LN^{-1}y_\iota), \end{aligned} \quad \text{(A.45)}$$

where $\bar{x}_\iota^u = \tilde{S}_\iota^{-1}(Y_\iota^{-1}\bar{x}_\iota + C_2^\intercal N^{-1}y_\iota)$. Now let us simplify U_t^*, where based on (A.39) and (A.41) we have

$$U_t^* = -\bar{\Upsilon}_t^{-1}\bar{N}_t\bar{X}_t^{-1}G_t(\bar{A}_\iota\bar{x}_\iota + \bar{D}_\iota\bar{y}_\iota). \quad \text{(A.46)}$$

Assume $l = t+1$, $\hat{K}_t := -\bar{\Upsilon}_t^{-1}\bar{N}_t$ and consider $\bar{\Upsilon}_t = \mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal$ where based on (A.36),

$$\mathcal{A} = \begin{bmatrix} \hat{R}_t & 0 \\ 0 & \bar{\Upsilon}_l - \bar{N}_l\bar{\Theta}_l^{-1}\bar{N}_l^\intercal \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} \hat{B}_t^\intercal \\ \bar{N}_l\bar{\Theta}_l^{-1} \end{bmatrix}, \ \ \mathcal{C} = \Xi_l.$$

Then, we have

$$\begin{aligned} \hat{K}_t = -\bar{\Upsilon}_t^{-1}\bar{N}_t = &-(\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal)^{-1}\mathcal{B}\mathcal{C}\bar{\bar{A}}_t \\ &-(\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal)^{-1}\begin{bmatrix} \bar{\bar{P}}_t^\intercal \\ 0 \end{bmatrix} = \hat{K}_t^1 + \hat{K}_t^2, \end{aligned}$$

where

$$\begin{aligned} &(\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal)^{-1}\mathcal{B}\mathcal{C} = \mathcal{A}^{-1}\mathcal{B}(\mathcal{C}^{-1} + \mathcal{B}^\intercal\mathcal{A}^{-1}\mathcal{B})^{-1} \\ &= \begin{bmatrix} \hat{R}_t^{-1}\hat{B}_t^\intercal \\ (\bar{\Upsilon}_l - \bar{N}_l\bar{\Theta}_l^{-1}\bar{N}_l^\intercal)^{-1}\bar{N}_l\bar{\Theta}_l^{-1} \end{bmatrix} (\bar{\Theta}_l^{-1} - \gamma^{-2}\hat{D}_t\hat{V}_t^{-1}\hat{D}_t^\intercal \\ &+ \hat{B}_t\hat{R}_t^{-1}\hat{B}_t^\intercal + \bar{\Theta}_l^{-1}\bar{N}_l^\intercal(\bar{\Upsilon}_l - \bar{N}_l\bar{\Theta}_l^{-1}\bar{N}_l^\intercal)^{-1}\bar{N}_l\bar{\Theta}_l^{-1}\big)^{-1}. \end{aligned}$$

Again the inequalities in (A.40) hold and by using them

$$\hat{K}_t^1 = -\begin{bmatrix} \hat{R}_t^{-1}\hat{B}_t^\intercal \\ \bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1} \end{bmatrix}\mathcal{S}_t^{-1}\bar{\bar{A}}_t.$$

Moreover,

$$\begin{aligned} &(\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{B}^\intercal)^{-1} = \big(I - \mathcal{A}^{-1}\mathcal{B}(\mathcal{C}^{-1} + \mathcal{B}^\intercal\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{B}^\intercal\big)\mathcal{A}^{-1} \\ &= \begin{bmatrix} I - \hat{R}_t^{-1}\hat{B}_t^\intercal\mathcal{S}_t^{-1}\hat{B}_t & -\hat{R}_t^{-1}\hat{B}_t^\intercal\mathcal{S}_t^{-1}\bar{\Theta}_t^{-1}\bar{N}_l^\intercal \\ -\bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1}\mathcal{S}_t^{-1}\hat{B}_t & I - \bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1}\mathcal{S}_t^{-1}\bar{\Theta}_t^{-1}\bar{N}_l^\intercal \end{bmatrix} \\ &\begin{bmatrix} \hat{R}_t^{-1} & 0 \\ 0 & (\bar{\Upsilon}_l - \bar{N}_l\bar{\Theta}_l^{-1}\bar{N}_l^\intercal)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I - \hat{R}_t^{-1}\hat{B}_t^\intercal\mathcal{S}_t^{-1}\hat{B}_t)\hat{R}_t^{-1} & \star \\ -\bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1}\mathcal{S}_t^{-1}\hat{B}_t\hat{R}_t^{-1} & \triangle \end{bmatrix}, \end{aligned}$$

and finally,

$$\hat{K}_t^2 = \begin{bmatrix} -(I - \hat{R}_t^{-1}\hat{B}_t^\intercal\mathcal{S}_t^{-1}\hat{B}_t) \\ \bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1}\mathcal{S}_t^{-1}\hat{B}_t \end{bmatrix}\hat{R}_l^{-1}\bar{\bar{P}}_t^\intercal.$$

Therefore,

$$\hat{K}_t = \hat{K}_t^1 + \hat{K}_t^2 = \begin{bmatrix} -\hat{R}_t^{-1}\hat{B}_t^\intercal\mathcal{S}_t^{-1}(\bar{\bar{A}}_t - \hat{B}_t\hat{R}_t^{-1}\bar{\bar{P}}_t^\intercal) - \hat{R}_t^{-1}\bar{\bar{P}}_t^\intercal \\ -\bar{\Upsilon}_l^{-1}\bar{N}_l\bar{X}_l^{-1}\mathcal{S}_t^{-1}(\bar{\bar{A}}_t - \hat{B}_t\hat{R}_t^{-1}\bar{\bar{P}}_t^\intercal) \end{bmatrix}.$$

Moreover, we have

$$\hat{R}_t^{-1}\bar{\bar{P}}_t^\intercal = (R + \gamma^{-2}P^\intercal\Pi_t^{-1}Y_tP)^{-1}P^\intercal\Pi_t^{-1} = R^{-1}P^\intercal\tilde{\Pi}_t^{-1},$$

and we can prove that $\bar{\bar{A}}_t - \hat{B}_s\hat{R}_t^{-1}\bar{\bar{P}}_t^\intercal = \bar{A}_t$. Furthermore, we can show that $\mathcal{S}_t^{-1}\bar{A}_t = X_lH_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1}$, therefore,

$$\bar{X}_l^{-1}\mathcal{S}_t^{-1}\bar{A}_t = (I - \gamma^{-2}Y_lX_l)H_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1}.$$

Then,

$$\hat{K}_t = -\begin{bmatrix} \hat{R}_t^{-1}\hat{B}_t^\intercal X_lH_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1} + R^{-1}P^\intercal\tilde{\Pi}_t^{-1} \\ \bar{\Upsilon}_l^{-1}\bar{N}_l(I - \gamma^{-2}Y_lX_l)H_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1} \end{bmatrix}.$$

Moreover, $\hat{R}_t^{-1}\hat{B}_t^\intercal = R^{-1}B_2^\intercal + \gamma^{-2}R^{-1}P^\intercal\tilde{\Pi}_t^{-1}Y_t\tilde{A}^\intercal$, and

$$\begin{aligned} &\gamma^{-2}R^{-1}P^\intercal\tilde{\Pi}_t^{-1}Y_t\tilde{A}^\intercal X_\ell H_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1} \\ &+ R^{-1}P^\intercal\tilde{\Pi}_t^{-1} = R^{-1}P^\intercal\tilde{\Pi}_t^{-1}\big(\gamma^{-2}Y_t(X_t - \tilde{Q}) + I \\ &- \gamma^{-2}Y_tX_t\big)(I - \gamma^{-2}Y_tX_t)^{-1} = R^{-1}P^\intercal(I - \gamma^{-2}Y_tX_t)^{-1}. \end{aligned}$$

Then by substitution into (A.46), it results in

$$\begin{aligned} U_t^* = -&\begin{bmatrix} R^{-1}(B_2^\intercal X_lH_t^{-1}\tilde{A} + P^\intercal)(I - \gamma^{-2}Y_tX_t)^{-1} \\ \bar{\Upsilon}_l^{-1}\bar{N}_l(I - \gamma^{-2}Y_lX_l)H_t^{-1}\tilde{A}(I - \gamma^{-2}Y_tX_t)^{-1} \end{bmatrix} \\ &\bar{X}_t^{-1}G_t(\check{A}\bar{x}_\iota^u + LN^{-1}y_\iota), \end{aligned}$$

where $\bar{x}_\iota^u = \tilde{S}_\iota^{-1}(C_2^\intercal N^{-1}y_\iota + Y_\iota^{-1}\bar{x}_\iota)$. By following the same procedure and decomposing $\bar{\Upsilon}_l^{-1}\bar{N}_l^\intercal$, we can rewrite the control policy based on a state estimator as follows

$$\begin{aligned} &\hat{x}_{s+1|s} = H_s^{-1}\tilde{A}\hat{x}_{s|s}, \\ &\quad\hat{x}_{s|s} = \begin{cases} X_s^{-1}G_s(\check{A}\bar{x}_{s-1}^u + LN^{-1}y_{s-1}), & \text{if } s = \tau\kappa + 1 \\ \hat{x}_{s|s-1}, & \text{otherwise,} \end{cases} \end{aligned} \quad \text{(A.47)}$$

where $s \in \mathbb{N}_0^{T-1}$. Then at every $s \neq \tau\kappa$, $u_s^* = K_s\hat{x}_{s|s}$, where

$$K_s = -R^{-1}(B_2^\intercal X_{s+1}H_s^{-1}\tilde{A} + P^\intercal),$$

and when $s = \tau\kappa$, then u_s^* follows (A.45). Moreover, based on (A.35), when $s \neq \kappa\tau \in \mathbb{N}_0^{T-1}$, $s \in \mathbb{N}_{\iota+1}^{\ell-2}$ and $t = s+1$, then

$$\hat{y}_s^* = (\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s)^{-1}\hat{D}_s^\intercal\bar{\Theta}_t\big(\bar{\bar{A}}_s\bar{x}_s + \begin{bmatrix} \hat{B}_s & \bar{\Theta}_t^{-1}\bar{N}_t^\intercal \end{bmatrix} U_s\big),$$

and when $s = \ell - 1$, then

$$\hat{y}_s^* := \big(\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{X}_\ell\hat{D}_s\big)^{-1}\hat{D}_s^\intercal\bar{X}_\ell(\bar{\bar{A}}_s\bar{x}_s + \hat{B}_su_s),$$

which can be simplified based on

$$\begin{aligned} &(\gamma^2\hat{V}_s - \hat{D}_s^\intercal\bar{\Theta}_t\hat{D}_s)^{-1}\hat{D}_s^\intercal\bar{\Theta}_t = \gamma^{-2}\hat{V}_s^{-1}\hat{D}_s^\intercal\Xi_t, \\ &\Xi_t = (\Theta_t^{-1} - \gamma^{-2}(B_1B_1^\intercal + A\Pi_s^{-1}Y_sA^\intercal))^{-1}, \\ &\Xi_t\bar{\bar{A}}_s = \Theta_tL_s^{-1}A(I - \gamma^{-2}Y_s\Theta_s)^{-1}, \\ &\hat{V}_s^{-1}\hat{D}_s^\intercal = C_2\Pi_s^{-1}Y_sA^\intercal + L^\intercal. \end{aligned}$$

Then for every $s \in \mathbb{N}_{\iota+1}^{\ell-1}$, $y_s^* = \tilde{\tilde{y}}_t + \check{\check{y}}_t + \hat{y}_t^*$. Moreover, based on (A.32), (A.36),

$$\bar{N}_{\ell-1} = \big(B_2^\intercal(X_\ell^{-1} - \gamma^{-2}B_1B_1^\intercal)^{-1}A + P^\intercal\big)\Omega_{\ell-1},$$

and for every $t \in \mathbb{N}_{\iota+1}^{\ell-2}$,

$$\bar{N}_t = \begin{bmatrix} B(\Theta_{t+1}^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A + P^\intercal \\ \bar{N}_{t+1} \bar{\Theta}_{t+1}^{-1} (\Theta_{t+1}^{-1} - \gamma^{-2} B_1 B_1^\intercal)^{-1} A \end{bmatrix} \Omega_t.$$

Furthermore, when $s = \kappa\tau = \iota$, then y_ι^* is determined in (A.44). Now following the similar procedure as in the proof of Lemma 2,

$$\begin{aligned} &\Delta W_\iota := W_\ell(\bar{x}_\ell) - W_\iota(\bar{x}_\iota) = W_\ell(\bar{x}_\ell) - \hat{W}_{\ell-1}(\bar{x}_{\ell-1}, u_{\ell-1}) \\ &+ \sum_{t=\iota+1}^{\ell-2} \left[\hat{W}_{t+1}(\bar{x}_{t+1}, U_{t+1}) - \hat{W}_t(\bar{x}_t, U_t)\right] \\ &+ \hat{W}_{\iota+1}(\bar{x}_{\iota+1}, U_{\iota+1}) - W_\iota(\bar{x}_\iota, \bar{y}_\iota) + W_\iota(\bar{x}_\iota, \bar{y}_\iota) - W_\iota(\bar{x}_\iota) \\ &= (U_\iota - U_\iota^*)^\intercal \Phi_\iota^\tau (U_\iota - U_\iota^*) - (\bar{Y}_\iota - \bar{Y}_\iota^*)^\intercal \Psi_\iota^\tau (\bar{Y}_\iota - \bar{Y}_\iota^*), \end{aligned}$$

for $\bar{Y}_\iota = [y_\iota^\intercal \dots y_{\ell-1}^\intercal]^\intercal$, $\bar{Y}_\iota^* = [y_\iota^{*\intercal} \dots y_{\ell-1}^{*\intercal}]^\intercal$

$$\Phi_\iota^\tau := \begin{bmatrix} \bar{R}_\iota + \bar{B}_\iota^\intercal \bar{\Theta}_{\iota+1} \bar{B}_\iota & \bar{B}_\iota^\intercal \bar{N}_{\iota+1}^\intercal \\ \bar{N}_{\iota+1} \bar{B}_\iota & \bar{\Upsilon}_{\iota+1} \end{bmatrix},$$

and $\Psi_\iota^\tau := \operatorname{diag}(\Psi_\iota, \dots, \Psi_{\iota+\tau-1})$, where for $t \in \mathbb{N}_{\iota+1}^{\ell-1}$, $\Psi_t := \gamma^2 \hat{V}_t - \hat{D}_t^\intercal \bar{\Theta}_{t+1} \hat{D}_t$ and $\Psi_\iota := \gamma^2 \bar{V}_\iota - \bar{D}_\iota^\intercal G_{\iota+1} \bar{D}_\iota$. Then by taking the summation over both sides of the above equation at all transmission times, we get

$$W_T(\bar{x}_T) - W_0(\bar{x}_0) = \textstyle\sum_{\kappa=0}^{\frac{T-1}{\tau}} \Delta W_{\kappa\tau}.$$

Therefore, similar to Lemma 2, we can show that

$$\textstyle\sum_{i=0}^{T-1} [z_i^\intercal z_i - \gamma^2 w_i^\intercal w_i] \leqslant -p_0(x_0) + \sum_{\kappa=0}^{\frac{T-1}{\tau}} \Delta W_{\kappa\tau}. \tag{A.48}$$

in the following, we will prove parts i and ii.

Part i) First we derive the conditions under which a τ-periodic ℓ_2-controller exists. Based on (A.29) and (A.34), we should have

$$\mathcal{C}_{h+1}^{-1} := \gamma^2 \hat{V}_h - \hat{D}_h^\intercal \bar{\Theta}_{h+1} \hat{D}_h \succ 0, \tag{A.49}$$

for every $h \in \mathbb{N}_1^{\tau-1}$, where according to (A.31) and (A.37), Θ_{h+1} is determined based on (32). We know $\hat{V}_h \succ 0$ and by assumming $\bar{\Theta}_{h+1} \succ 0$, for all $h \in \mathbb{N}_1^{\tau-1}$, we have the following equality based on the matrix inversion lemma

$$\mathcal{C}_{h+1} = \gamma^{-2} \hat{V}_h^{-1} + \gamma^{-4} \hat{V}_h^{-1} \hat{D}_h^\intercal \Xi_{h+1} \hat{D}_h \hat{V}_h^{-1},$$

where $\Xi_{h+1} = (\bar{\Theta}_{h+1}^{-1} - \gamma^{-2} \hat{D}_h \hat{V}_h^{-1} \hat{D}_h^\intercal)^{-1}$. This shows that $\mathcal{C}_{h+1}^{-1}$ is positive definite if Ξ_{h+1} is positive definite, assuming that $\bar{\Theta}_{h+1} \succ 0$. Furthermore, again by using the matrix inversion lemma and assumming $\bar{\Theta}_{h+1} \succ 0$, for all $h \in \mathbb{N}_1^{\tau-1}$, we can rewrite Ξ_{h+1} as follows

$$\Xi_{h+1} = \bar{\Theta}_{h+1} + \bar{\Theta}_{h+1} \hat{D}_h \mathcal{C}_{h+1} \hat{D}_h^\intercal \bar{\Theta}_{h+1}.$$

This shows that Ξ_{h+1} is positive definite if $\mathcal{C}_{h+1}^{-1} \succ 0$, assuming that $\bar{\Theta}_{h+1} \succ 0$. Therefore, the condition (A.49) is equivalent to $\Xi_{h+1}^{-1} \succ 0$, for every $h \in \mathbb{N}_1^{\tau-1}$, if $\bar{\Theta}_{h+1} \succ 0$. Then according to (A.38), we should have

$$\Xi_{h+1}^{-1} = \Theta_{h+1}^{-1} - \gamma^{-2} (A \Pi_h^{-1} Y_h A^\intercal + B_1 B_1^\intercal) \succ 0, \tag{A.50}$$

for all $h \in \mathbb{N}_1^{\tau-1}$. On the other hand, $\bar{\Theta}_{h+1}^{-1} = \Theta_{h+1}^{-1} - \gamma^{-2} Y_{h+1}$. We can easily conclude that $A \Pi_h^{-1} Y_h A^\intercal + B_1 B_1^\intercal \geqslant Y_{h+1}$. Therefore, satisfiying (A.50) will automatically satisfy $\bar{\Theta}_{h+1} \succ 0$, for all $h \in \mathbb{N}_1^{\tau-1}$. Moreover, for the infinite-horizon problem, the Riccati equations (A.43) and (9) converge to stationary positive definite matrices X and Y, respectively, for which the conditions (A.18), (A.19), and (A.20) are required, following the similar reasoning as in Lemma 2.

Then according to (A.51), we have

$$J = \textstyle\sum_{i=0}^{\infty} [z_i^\intercal z_i - \gamma^2 w_i^\intercal w_i] \leqslant \gamma^2 x_0^\intercal Y^{-1} x_0 + \sum_{\kappa=0}^{\infty} \Delta \bar{W}_{\kappa\tau}, \tag{A.51}$$

where

$$\Delta \bar{W}_{\kappa\tau} = (U_\iota - U_\iota^*)^\intercal \bar{\Phi}_\tau (U_\iota - U_\iota^*) - (\bar{Y}_\iota - \bar{Y}_\iota^*)^\intercal \bar{\Psi}_\tau (\bar{Y}_\iota - \bar{Y}_\iota^*),$$

in which U_ι^* and $\bar{Y}_\iota^*$ are the stationary version of the control and output policies, respectively, which are introduced in the Lemma. Now for $U_\iota = U_\iota^*$ at all $\iota \in \mathbb{N}_0$, we can easily conclude $J \leqslant \gamma^2 x_0^\intercal Y^{-1} x_0$.

Global Asymptotic stability: Similar to Lemma 2, having $\Omega_\tau = (I - \gamma^{-2} Y X)^{-1}$, $W_{\kappa\tau}(\bar{x}_{\kappa\tau}) = \bar{x}_{\kappa\tau}^\intercal X \Omega_\tau \bar{x}_{\kappa\tau} + Z_{\kappa\tau}$ and $p_{\kappa\tau}(x_{\kappa\tau}) = -\gamma^2 (x_{\kappa\tau} - \bar{x}_{\kappa\tau})^\intercal Y^{-1} (x_{\kappa\tau} - \bar{x}_{\kappa\tau}) + Z_{\kappa\tau}$, then

$$W_{\kappa\tau}(\bar{x}_{\kappa\tau}) - p_{\kappa\tau}(x_{\kappa\tau}) - W_0(\bar{x}_0) + p_0(x_0) \leqslant \textstyle\sum_{i=0}^{\kappa-1} \Delta \bar{W}_{i\tau}. \tag{A.52}$$

Now we take $V(x_{\kappa\tau}, \bar{x}_{\kappa\tau}) := -p_{\kappa\tau}(x_{\kappa\tau}) + W_{\kappa\tau}(\bar{x}_{\kappa\tau})$ as the Lyapunov function candidate for every $\kappa \in \mathbb{N}_0$. Then based on (A.52), we have

$$\Delta V(x_{\kappa\tau}, \bar{x}_{\kappa\tau}) := V(x_{\kappa\tau}, \bar{x}_{\kappa\tau}) - V(x_0, \bar{x}_0) \leqslant \textstyle\sum_{i=0}^{\kappa-1} \Delta \bar{W}_{i\tau}. \tag{A.53}$$

Now if $U_\iota = U_\iota^*$ at all $\iota \in \mathbb{N}_0$, then $\Delta V(x_{\kappa\tau}, \bar{x}_{\kappa\tau}) \leqslant 0$ for all $\kappa \in \mathbb{N}_0$, which proves that the control loop is Lyapunov stable. Following the similar arguments as in the proof of Lemma 2, we can show that $x_t \to 0$ as time goes to infinity, when $w = (0, 0, \dots)$ and $U_\iota = U_\iota^*$ at all $\iota \in \mathbb{N}_0$.

Part ii) This is already proved in (A.51).

D Proof of Theorem 1

Based on (28), to show that the scheduling and control policy (47), (43) is an ℓ_2-controller with an ℓ_2-gain bound γ, firstly, we have to show that for any realization of y_t, $t \in \mathbb{N}_0$,

$$\sum_{t=0}^{\infty} (\mathcal{R}_t^\mu - u_t^*)^\intercal \Phi (\mathcal{R}_t^\mu - u_t^*) - (y_t - y_t^*)^\intercal \Psi (y_t - y_t^*) \leqslant 0, \tag{A.54}$$

where at every $t \in \mathbb{N}_0$, $\mathcal{R}_t^\mu$ follows (43) and u_t^*, y_t^* follow (22), (29), respectively. According to the scheduling law (47), for every $t \in \mathbb{N}$,

$$\beta_{t+1|t+1} = \sum_{i=0}^{t} (\mathcal{R}_i^\mu - u_i^*)^\intercal \Psi (\mathcal{R}_i^\mu - u_i^*) - (y_i - y_i^*)^\intercal \Phi (y_i - y_i^*) \leqslant 0.$$

Therefore, (A.54) also holds. On the other hand, based on (28) and (A.54), we conclude $J \leqslant \gamma^2 x_0^\intercal Y^{-1} x_0$.

Now we need to prove the Lyapunov stability of the proposed event-triggered control policy. The inequality (A.23) also holds in this setting for all $t \in \mathbb{N}$ and $\sum_{i=0}^{t-1} \Delta \bar{W}_i = \beta_{t|t} \leqslant 0$. This results in $V(x_t, \bar{x}_t) - V(x_0, \bar{x}_0) \leqslant 0$ for all $t \in \mathbb{N}_0$, which guarantees the Lyapunov stability of the control loop. Moreover, similar to the proof of Lemma 2, we can show that for $w = (0, 0, \dots)$ and the proposed ETC, the state converges to zero as time goes to infinity. Therefore, the control loop is globally asymptotically stable.

E Proof of Theorem 2

The proof is similar to what we presented in the proof of Theorem 1, however, we just need to consider (35) and (A.53).

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Complementary Material: Proof of Lemma 1

To solve the optimization problem in (6), we follow a dynamic programming approach, starting from the initial time and moving forward. By assumption, we have $p_0(x_0) = -\gamma^2 x_0^\intercal Y_0^{-1} x_0$. Now to find $p_1(x_1)$, consider $x := x_1$, $\zeta := x_0$, $y := y_0$, $u := u_0$ and $w := w_0$. Then,

$$p_1(x) = \max_{w \in \mathbb{R}^d} \max_{\zeta \in \mathbb{R}^n} \big\{ p_0(\zeta) + \zeta^\intercal Q \zeta + u^\intercal R u + 2\zeta^\intercal P u \\ - \gamma^2 w^\intercal w;\ y = C_2 \zeta + D_{21} w,\ x = A\zeta + B_1 w + B_2 u \big\}.$$

Based on the dynamics of y, we can parameterize w as

$$w = D_{21}^\intercal N^{-1}(y - C_2 \zeta) + D\nu, \tag{A.55}$$

for an arbitrary $\nu \in \mathbb{R}^d$ and $D := I - D_{21}^\intercal N^{-1} D_{21}$. Then we can substitute the above equality into the state dynamics, which results in

$$x = A\zeta + LN^{-1}(y - C_2\zeta) + B_1 D\nu + B_2 u. \tag{A.56}$$

Moreover, we have

$$\beta_0(\zeta, u, y, \nu) := -\zeta^\intercal(\gamma^2 Y_0^{-1} - Q)\zeta + u^\intercal R u + 2\zeta^\intercal P u \\ -\gamma^2 w^\intercal w = -\gamma^2 \zeta^\intercal S_0 \zeta + 2\gamma^2 \zeta^\intercal (C_2^\intercal N^{-1} y + \gamma^{-2} P u) \\ + u^\intercal R u - \gamma^2 \nu^\intercal D \nu - \gamma^2 y^\intercal N^{-1} y,$$

where $S_0 := Y_0^{-1} + C_2^\intercal N^{-1} C_2 - \gamma^{-2} Q$. Assuming S_0 to be invertible, then

$$\beta_0(\zeta, u, y, \nu) = -\gamma^2 (\zeta - \zeta^*)^\intercal S_0 (\zeta - \zeta^*) + u^\intercal R u \\ - \gamma^2 \nu^\intercal D \nu - \gamma^2 y^\intercal N^{-1} y + \gamma^2 \zeta^{\intercal *} S_0 \zeta^*, \tag{A.57}$$

for $\zeta^* := S_0^{-1}(C_2^\intercal N^{-1} y + \gamma^{-2} P u)$. Moreover, based on (A.56),

$$\zeta = \hat{A}^\dagger b + (I - \hat{A}^\dagger \hat{A})\xi, \tag{A.58}$$

where $\hat{A}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\hat{A}$, $\xi \in \mathbb{R}^n$ is an arbitrary vector, and

$$b = x - B_2 u - LN^{-1} y - B_1 D\nu.$$

Then, by substituting (A.58) into (A.57), we get

$$\begin{aligned}
&\chi_0(\xi, u, y, \nu) := -\gamma^2 \big(\hat{A}^{\dagger} b + (I - \hat{A}^{\dagger}\hat{A})\xi - \zeta^*\big)^{\intercal} S_0 \\
&\quad \big(\hat{A}^{\dagger} b + (I - \hat{A}^{\dagger}\hat{A})\xi - \zeta^*\big) - \gamma^2 \nu^{\intercal} D\nu \\
&\quad + u^{\intercal} Ru - \gamma^2 y^{\intercal} N^{-1} y + \gamma^2 \zeta^{\intercal *} S_0 \zeta^*.
\end{aligned}$$

Now the maximization of $\beta_0(\zeta, u, y, \nu)$ with respect to ζ is equivalent to the maximization of $\chi_0(\xi, u, y, \nu)$ with respect to $\xi \in \mathbb{R}^n$. First, let us replace ζ^* by $\hat{A}^{\dagger}\hat{A}\zeta^* + (I - \hat{A}^{\dagger}\hat{A})\zeta^*$ in the above equation

$$\begin{aligned}
&\chi_0(\xi, u, y, \nu) = -\gamma^2 \big(\hat{A}^{\dagger}(b - \hat{A}\zeta^*) + (I - \hat{A}^{\dagger}\hat{A})(\xi - \zeta^*)\big)^{\intercal} S_0 \\
&\quad \big(\hat{A}^{\dagger}(b - \hat{A}\zeta^*) + (I - \hat{A}^{\dagger}\hat{A})(\xi - \zeta^*)\big) + u^{\intercal} Ru \\
&\quad - \gamma^2 y^{\intercal} N^{-1} y - \gamma^2 \nu^{\intercal} D\nu + \gamma^2 \zeta^{\intercal *} S_0 \zeta^*.
\end{aligned}$$

Since $\hat{A}(I - \hat{A}^{\dagger}\hat{A}) = 0$, if $S_0 \succ 0$, then

$$\arg\max\nolimits_{\xi \in \mathbb{R}^n} \chi_0(\xi, u, y, \nu) = \zeta^*,$$

and

$$\begin{aligned}
&\kappa_0(x, u, y, \nu) := \max\nolimits_{\xi \in \mathbb{R}^n} \chi_0(\xi, u, y, \nu) \\
&\quad = -\gamma^2 \big(x - B_2 u - LN^{-1} y - B_1 D\nu - \hat{A}\zeta^*\big)^{\intercal} \hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} \\
&\quad \big(x - B_2 u - LN^{-1} y - B_1 D\nu - \hat{A}\zeta^*\big) - \gamma^2 \nu^{\intercal} D\nu \\
&\quad + u^{\intercal} Ru - \gamma^2 y^{\intercal} N^{-1} y + \gamma^2 \zeta^{\intercal *} S_0 \zeta^*.
\end{aligned}$$

Note that $\max_{\xi \in \mathbb{R}^n} \chi_0(\xi, u, y, \nu)$ is unbounded if S_0 has a negative eigenvalue. Now to determine $p_1(x)$, we just need to solve the following optimization problem

$$p_1(x) = \max\nolimits_{\nu \in \mathbb{R}^d} \kappa_0(x, u, y, \nu). \tag{A.59}$$

By taking the derivative of $\kappa_0(x, u, y, \nu)$ with respect to ν and equating it to zero, we get

$$D(\Omega_0 \nu^* - c) = 0,$$

where

$$\begin{aligned}
&\Omega_0 = I + B_1^{\intercal} \hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} B_1 D, \\
&c = B_1^{\intercal} \hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} (x - B_2 u - LN^{-1} y - \hat{A}\zeta^*), \\
&\nu^* = \arg\max\nolimits_{\nu \in \mathbb{R}^d} \kappa_0(x, u, y, \nu).
\end{aligned}$$

We know that $\alpha D D_{12}^{\intercal} N^{-1} = 0$, for any $\alpha \in \mathbb{R}$, therefore,

$$\Omega_0 \nu^* - c = \alpha D_{12}^{\intercal} N^{-1}.$$

Accordingly, assuming $\alpha = 0$, one value for ν^* is

$$\nu^* := \Omega_0^{-1} B_1^{\intercal} \hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} (x - B_2 u - LN^{-1} y - \hat{A}\zeta^*),$$

given the condition that $\Omega_0 \succ 0$. By using the matrix inversion lemma and considering

$$Y_1 := \hat{A} S_0^{-1} \hat{A}^{\intercal} + B_1 D B_1^{\intercal}, \tag{A.60}$$

then under the condition that $S_0 \succ 0$ and $Y_1 \succ 0$, we have

$$\nu^* := B_1^{\intercal} Y_1^{-1} (x - B_2 u - LN^{-1} y - \hat{A}\zeta^*).$$

Moreover, by using the matrix inversion lemma, we have

$$\Omega_0^{-1} = I - D B_1^{\intercal} (\hat{A} S_0^{-1} \hat{A}^{\intercal} + B_1 D B_1^{\intercal})^{-1} B_1.$$

The above equality indicates that $S_0 \succ 0$ is enough to guarantee $\Omega_0 \succ 0$. Therefore, if $S_0 \succ 0$, then

$$\begin{aligned}
&p_1(x) = \kappa_0(x, u, y, \nu^*) = -\gamma^2 (x - B_2 u - LN^{-1} y - \hat{A}\zeta^*)^{\intercal} \\
&\quad \hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} \big(I - B_1 D B_1^{\intercal} Y_1^{-1}\big)(x - B_2 u - LN^{-1} y - \hat{A}\zeta^*) \\
&\quad + u^{\intercal} Ru - \gamma^2 y^{\intercal} N^{-1} y + \gamma^2 \zeta^{\intercal *} S_0 \zeta^*.
\end{aligned}$$

Accordingly, by using (A.60), we can conclude

$$\hat{A}^{\dagger\intercal} S_0 \hat{A}^{\dagger} \big(I - B_1 D B_1^{\intercal} Y_1^{-1}\big) = Y_1^{-1},$$

which finally simplifies $p_1(x)$ as follows

$$p_1(x) = -\gamma^2 (x - \bar{x}_1)^{\intercal} Y_1^{-1} (x - \bar{x}_1) + Z_1,$$

under the condition that $S_0 \succ 0$, and $Y_1 \succ 0$, where

$$\begin{aligned}
&\bar{x}_1 := (\hat{A} S_0^{-1} C_2^{\intercal} + L) N^{-1} y + (B_2 + \gamma^{-2} \hat{A} S_0^{-1} P) u, \\
&Z_1 := u^{\intercal} (R + \gamma^{-2} P^{\intercal} S_0^{-1} P) u + 2 y^{\intercal} N^{-1} C_2 S_0^{-1} P u \\
&\quad - \gamma^2 y^{\intercal} N^{-1} (I - C_2 S_0^{-1} C_2^{\intercal} N^{-1}) y.
\end{aligned}$$

We can also show that the controllability of $(\hat{A}, B_1 D)$ and $S_0 \succ 0$ are sufficient conditions to guarantee $Y_1 \succ 0$. Based on (A.60), we can infer that Y_1 is at least positive semi-definite when $S_0 \succ 0$. Now let us take $\varrho \in \mathbb{R}^n$ as a nonzero eigenvector of Y_1 corresponding to its possible zero eigenvalue. Then based on (A.60), we have

$$\varrho^{\intercal} \hat{A} = 0, \quad \varrho^{\intercal} B_1 D = 0.$$

This result in

$$\varrho^{\intercal} [B_1 D \quad \hat{A} B_1 D \ldots \hat{A}^{n-1} B_1 D] = 0.$$

Therefore, the controllability matrix for $(\hat{A}, B_1 D)$ does not have n independent rows, which contradicts the controllability assumption. Thus, Y_1 is positive definite.

Now, based on an induction argument, we assume that at an arbitrary time $t \in \mathbb{N}_0$,

$$p_t(x_t) = -\gamma^2 (x_t - \bar{x}_t)^{\intercal} Y_t^{-1} (x_t - \bar{x}_t) + Z_t, \tag{A.61}$$

where $\bar{x}_t$ is a known state estimation. Then in order to find the information state at time $t+1$, we need to solve

$$\begin{aligned}
&p_{t+1}(x) = \max_{w \in \mathbb{R}^d} \max_{\zeta \in \mathbb{R}^n} \{p_t(\zeta) + \zeta^{\intercal} Q \zeta + u^{\intercal} R u + 2 \zeta^{\intercal} P u \\
&\quad - \gamma^2 w^{\intercal} w;\ y = C_2 \zeta + D_{21} w,\ x = A\zeta + B_1 w + B_2 u\},
\end{aligned}$$

where $x := x_{t+1}$, $\zeta := x_t$, $y := y_t$, $u := u_t$, $w := w_t$. We follow the same steps as before and firstly substitute the disturbance vector (A.55) into the optimization cost function. Therefore, assuming $\hat{x} := \bar{x}_t$, we have

$$\begin{aligned}
&\beta_t(\zeta, u, y, \nu) := -\gamma^2 (\zeta - \hat{x})^{\intercal} Y_t^{-1} (\zeta - \hat{x}) + Z_t + \zeta^{\intercal} Q \zeta \\
&\quad + u^{\intercal} R u + 2 \zeta^{\intercal} P u - \gamma^2 w^{\intercal} w = -\gamma^2 (\zeta^{\intercal} S_t \zeta - 2 \zeta^{\intercal} \check{R}_t) \\
&\quad + Z_t + u^{\intercal} R u - \gamma^2 \nu^{\intercal} D \nu - \gamma^2 (y^{\intercal} N^{-1} y + \hat{x}^{\intercal} Y_t^{-1} \hat{x}),
\end{aligned}$$

where

$$\begin{aligned}
&S_t := Y_t^{-1} + C_2^{\intercal} N^{-1} C_2 - \gamma^{-2} Q, \\
&\check{R}_t := C_2^{\intercal} N^{-1} y + Y_t^{-1} \hat{x} + \gamma^{-2} P u.
\end{aligned}$$

We assume S_t is invertible, and complete the squares for ζ as

$$\beta_t(\zeta, u, y, \nu) = -\gamma^2(\zeta - \zeta^*)^\intercal S_t(\zeta - \zeta^*) + \gamma^2\zeta^{*\intercal} S_t\zeta^* + Z_t \\ + u^\intercal Ru - \gamma^2\nu^\intercal D\nu - \gamma^2(y^\intercal N^{-1}y + \hat{x}^\intercal Y_t^{-1}\hat{x}),$$

where $\zeta^* := S_t^{-1}\check{R}_t$. Then by substituting (A.58) into the above equation, doing maximization with respect to ξ and rewriting it based on x, we get

$$\kappa_t(x, u, y, \nu) := -\gamma^2\big(x - B_2u - LN^{-1}y - B_1D\nu - \hat{A}\zeta^*\big)^\intercal \\ \hat{A}^{\dagger\intercal} S_t\hat{A}^\dagger\big(x - B_2u - LN^{-1}y - B_1D\nu - \hat{A}\zeta^*\big) - \gamma^2\nu^\intercal D\nu \\ + \gamma^2\zeta^{*\intercal} S_t\zeta^* + Z_t + u^\intercal Ru - \gamma^2(y^\intercal N^{-1}y + \hat{x}^\intercal Y_t^{-1}\hat{x}),$$

under the condition that $S_t \succ 0$ (if S_t has a negative eigenvalue, then the maximization with respect to ξ does not have a solution). Finally,

$$p_{t+1}(x) = \max\nolimits_{\nu\in\mathbb{R}^d} \kappa_t(x, u, y, \nu), \tag{A.62}$$

which results in

$$\nu^* = \Omega_t^{-1}B_1^\intercal\hat{A}^{\dagger\intercal} S_t\hat{A}^\dagger(x - B_2u - LN^{-1}y - \hat{A}\zeta^*),$$

where $\Omega_t = I + B_1^\intercal\hat{A}^{\dagger\intercal} S_t\hat{A}^\dagger B_1D$, given the condition that $\Omega_t \succ 0$ (which is equivalent to $S_t \succ 0$ following the similar steps as before). Therefore, $p_{t+1}(x) = \kappa_t(x, u, y, \nu^*)$. By following the similar simplification approach as before, and changing $x, \zeta, y, u, \hat{x}$ to $x_{t+1}, x_t, y_t, u_t, \bar{x}_t$, respectively, we get

$$p_{t+1}(x_{t+1}) = -\gamma^2(x_{t+1} - \bar{x}_{t+1})^\intercal Y_{t+1}^{-1}(x_{t+1} - \bar{x}_{t+1}) + Z_{t+1},$$

under the condition that $S_t \succ 0$ and $Y_{t+1} \succ 0$ (which holds based on the controllability of $(\hat{A}, B_1D)$ and $S_t \succ 0$), where

$$Y_{t+1} = \hat{A}S_t^{-1}\hat{A}^\intercal + B_1DB_1^\intercal, \tag{A.63}$$

and

$$Z_{t+1} = Z_t + u_t^\intercal(R + \gamma^{-2}P^\intercal S_t^{-1}P)u_t + 2y_t^\intercal N^{-1}C_2S_t^{-1}Pu_t \\ - \gamma^2y_t^\intercal N^{-1}(I - C_2S_t^{-1}C_2^\intercal N^{-1})y_t + 2u_t^\intercal P^\intercal S_t^{-1}Y_t^{-1}\bar{x}_t \\ - \gamma^2\bar{x}_t^\intercal Y_t^{-1}(I - S_t^{-1}Y_t^{-1})\bar{x}_t + 2\gamma^2y_t^\intercal N^{-1}C_2S_t^{-1}Y_t^{-1}\bar{x}_t.$$

Furthermore, the state estimator is

$$\bar{x}_{t+1} = \hat{A}\zeta^* + B_2u_t + LN^{-1}y_t = \hat{A}\bar{x}_t + B_2u_t + LN^{-1}y_t \\ + \hat{A}S_t^{-1}\big(\gamma^{-2}Q\bar{x}_t + \gamma^{-2}Pu_t + C_2^\intercal N^{-1}(y_t - C_2\bar{x}_t)\big).$$

Since the information state at time $t + 1$ has the same form as the one assumed at time t, then the assumption of the induction is true and the information state follows (A.61) if $S_t \succ 0$ for every $t \in \mathbb{N}_0$, where Y_t follows (A.63) and $Z_t,\ \bar{x}_t$ are determined based on the previous two equation for $Z_0 = 0,\ \bar{x}_0 = 0$.

In order to simplify the future analysis, we can rewrite Z_{t+1} in such a way that no coupling terms are present. We carry out this first for the coupling term between y_t and u_t. Consider $\bar{R}_t := R + \gamma^{-2}P^\intercal S_t^{-1}P$ and $\tilde{S}_t = Y_t^{-1} + C_2^\intercal N^{-1}C_2 - \gamma^{-2}\tilde{Q}$, then

$$\check{u}_t : = -\bar{R}_t^{-1}P^\intercal S_t^{-1}C_2^\intercal N^{-1}y_t = -R^{-1}P^\intercal\tilde{S}_t^{-1}C_2^\intercal N^{-1}y_t$$

Therefore,

$$Z_{t+1} = Z_t + (u_t - \check{u}_t)^\intercal\bar{R}_t(u_t - \check{u}_t) + 2(u_t - \check{u}_t)^\intercal P^\intercal S_t^{-1}Y_t^{-1}\bar{x}_t \\ - \gamma^2y_t^\intercal N^{-1}(I - C_2S_t^{-1}C_2^\intercal N^{-1})y_t - \check{u}_t^\intercal\bar{R}_t\check{u}_t + 2\check{u}_t^\intercal P^\intercal S_t^{-1}Y_t^{-1} \\ \bar{x}_t - \gamma^2\bar{x}_t^\intercal Y_t^{-1}(I - S_t^{-1}Y_t^{-1})\bar{x}_t + 2\gamma^2y_t^\intercal N^{-1}C_2S_t^{-1}Y_t^{-1}\bar{x}_t. \tag{A.64}$$

Now by substituting $\check{u}_t$ into the above equation, considering $\hat{u}_t := u_t - \check{u}_t$ and noticing that

$$N^{-1}C_2\tilde{S}_t^{-1}PR^{-1}P^\intercal S_t^{-1}C_2^\intercal N^{-1} - \gamma^2N^{-1}(C_2S_t^{-1}C_2^\intercal N^{-1} \\ - I) = \gamma^2N^{-1}(I - C_2\tilde{S}_t^{-1}C_2^\intercal N^{-1}), \\ - N^{-1}C_2\tilde{S}_t^{-1}PR^{-1}P^\intercal S_t^{-1}Y_t^{-1} + \gamma^2N^{-1}C_2S_t^{-1}Y_t^{-1} \\ = \gamma^2N^{-1}C_2\tilde{S}_t^{-1}Y_t^{-1},$$

we can simplify Z_{t+1} as follows

$$Z_{t+1} = Z_t + \hat{u}_t^\intercal\bar{R}_t\hat{u}_t - \gamma^2y_t^\intercal\bar{V}_ty_t - \gamma^2\bar{x}_t^\intercal Y_t^{-1}(I - S_t^{-1}Y_t^{-1})\bar{x}_t \\ + 2\hat{u}_t^\intercal P^\intercal S_t^{-1}Y_t^{-1}\bar{x}_t + 2\gamma^2y_t^\intercal N^{-1}C_2\tilde{S}_t^{-1}Y_t^{-1}\bar{x}_t,$$

where

$$\bar{V}_t := N^{-1}(I - C_2\tilde{S}_t^{-1}C_2^\intercal N^{-1}). \tag{A.65}$$

Then by considering $\bar{u}_t := \hat{u}_t - \tilde{u}_t$, $\bar{y}_t := y_t - \tilde{y}_t$, we get

$$Z_{t+1} = Z_t + \bar{u}_t^\intercal\bar{R}_t\bar{u}_t - \gamma^2\bar{y}_t^\intercal\bar{V}_t\bar{y}_t + \bar{x}_t^\intercal\bar{Q}_t\bar{x}_t,$$

where

$$\tilde{u}_t := -\bar{R}_t^{-1}P^\intercal S_t^{-1}Y_t^{-1}\bar{x}_t = -R^{-1}P^\intercal\tilde{S}_t^{-1}Y_t^{-1}\bar{x}_t, \\ \tilde{y}_t := \bar{V}_t^{-1}N^{-1}C_2\tilde{S}_t^{-1}Y_t^{-1}\bar{x}_t = C_2\tilde{\Pi}_t^{-1}\bar{x}_t,$$

for $\tilde{\Pi}_t := I - \gamma^{-2}Y_t\tilde{Q}$. Moreover,

$$\bar{Q}_t := \gamma^2(Y_t^{-1}S_t^{-1}Y_t^{-1} - Y_t^{-1}) - Y_t^{-1}S_t^{-1}PR^{-1}P^\intercal\tilde{S}_t^{-1}Y_t^{-1} \\ + \gamma^2Y_t^{-1}\tilde{S}_t^{-1}C_2^\intercal N^{-1}C_2\tilde{\Pi}_t^{-1} = \tilde{Q}\tilde{\Pi}_t^{-1}.$$

Finally, we can represent the state estimator as follows

$$\bar{x}_{t+1} = \hat{A}\bar{x}_t + B_2\hat{u}_t + \big(L + (\hat{A} - B_2R^{-1}P^\intercal)\tilde{S}^{-1}C_2^\intercal\big)N^{-1}y_t \\ + \hat{A}S_t^{-1}(\gamma^{-2}Q\bar{x}_t + \gamma^{-2}P\hat{u}_t - C_2^\intercal N^{-1}C_2\bar{x}_t) = \hat{A}\bar{x}_t + B_2\bar{u}_t \\ + B_2\tilde{u}_t + (L + \check{A}\tilde{S}^{-1}C_2^\intercal)N^{-1}\bar{y}_t + (L + \check{A}\tilde{S}^{-1}C_2^\intercal)N^{-1}\tilde{y}_t \\ + \hat{A}S_t^{-1}(\gamma^{-2}Q\bar{x}_t + \gamma^{-2}P\bar{u}_t + \gamma^{-2}P\tilde{u}_t - C_2^\intercal N^{-1}C_2\bar{x}_t) \\ = \bar{A}_t\bar{x}_t + \bar{B}_t\bar{u}_t + \bar{D}_t\bar{y}_t,$$

where $\bar{A}_t := \tilde{A}\tilde{\Pi}_t^{-1}$, $\bar{B}_t := B_2 + \gamma^{-2}\hat{A}S_t^{-1}P$, and

$$\bar{D}_t := \big(L + (\hat{A} - B_2R^{-1}P^\intercal)\tilde{S}_t^{-1}C_2^\intercal\big)N^{-1} = \tilde{A}\tilde{S}_t^{-1}C_2^\intercal N^{-1} + L\bar{V}_t.$$